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# CIE A-LEVEL FURTHER MATHS 9231 (FP)

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FORMULAE & SOLVED QUESTIONS FOR FURTHER PURE

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## 1. ROOTS OF EQUATIONS

### 1.1 Coefficients of Equations

	Sum of Products of Roots			
	1	2	3	4
$x^2 + bx + c = 0$	$-b$	$c$	$\frac{b^2 - 4c}{4}$	$\frac{b^3 - 6bc + 8c^2}{8}$
$x^3 + bx^2 + cx + d = 0$	$-b$	$c$	$-d$	$\frac{b^3 - 6bc + 8c^2}{8}$
$x^4 + bx^3 + cx^2 + dx + e = 0$	$-b$	$c$	$-d$	$e$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

### 1.2 Algebraic Combinations

- Finding an equation through algebraic manipulation

#### Ex 1.3

#### Question 2b:

$x^3 + 3x^2 - 2x + 5 = 0$  has roots  $\alpha, \beta, \gamma$

Find equation with roots  $(\alpha - 1), (\beta - 1), (\gamma - 1)$

Solution:

Using coefficients:

- $\alpha + \beta + \gamma = -3$
- $\alpha\beta + \beta\gamma + \alpha\gamma = -2$
- $\alpha\beta\gamma = -5$

Equation to find:

$$1. \quad (\alpha - 1) + (\beta - 1) + (\gamma - 1) = \\ = \alpha + \beta + \gamma - 3$$

$$\therefore -3 - 3 = 6$$

$$2. \quad (\alpha - 1)(\beta - 1) + (\alpha - 1)(\gamma - 1) + (\beta - 1)(\gamma - 1)$$

$$= \alpha\beta - \alpha - \beta + 1 + \alpha\gamma - \alpha - \gamma + 1 + \beta\gamma - \beta - \gamma + 1$$

$$= (\alpha\beta + \alpha\gamma + \beta\gamma) - 2(\alpha + \beta + \gamma) + 3$$

$$\therefore -2 - 2(-3) + 3 = 7$$

$$3. \quad (\alpha - 1)(\beta - 1)(\gamma - 1)$$

$$= (\alpha\beta - \alpha - \beta + 1)(\gamma - 1)$$

$$= \alpha\beta\gamma - \alpha\beta - \alpha\gamma + \alpha - \beta\gamma + \beta + \gamma - 1$$

$$= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) - 1$$

$$\therefore -5 - (-2) + (-3) + 1 = -7$$

Thus equation is:  $x^3 + 6x^2 + 7x + 7 = 0$

### 1.3 Substitution

- For finding equations whose roots have been raised by a power or undergone common algebraic manipulation

#### {S03-P01}

#### Question 5:

$8x^3 + 12x^2 + 4x - 1 = 0$  has roots  $\alpha, \beta, \gamma$

Find equation with roots  $(2\alpha + 1), (2\beta + 1), (2\gamma + 1)$

Solution:

$$\text{Let } y = 2x + 1 \quad \therefore x = \frac{y-1}{2}$$

Substitute into equation:

$$8\left(\frac{y-1}{2}\right)^3 + 12\left(\frac{y-1}{2}\right)^2 + 4\left(\frac{y-1}{2}\right) - 1 = 0$$

$$\frac{8(y-1)^3}{8} + \frac{12(y-1)^2}{4} + \frac{4(y-1)}{2} - 1 = 0$$

$$(y-1)^3 + 3(y-1)^2 + 2(y-1) - 1 = 0$$

$$(y-1)(y^2 - 2y + 1) + 3(y^2 - 2y + 1) + 2y - 2 - 1 = 0$$

$$y^3 - 2y^2 + y - y^2 + 2y - 1 + 3y^2 - 6y + 3 + 2y - 2 - 1 = 0$$

$$y^3 - y - 1 = 0$$

Thus equation is:  $x^3 - x - 1 = 0$

### 1.4 Recurrence Relations

- For finding sums of roots to a specific degree of power

#### {S03-P01}

#### Question 5:

Following the equation found above, find  $S_{-2}$  where

$$S_n = (2\alpha + 1)^n + (2\beta + 1)^n + (2\gamma + 1)^n$$

Solution:

$x^3 - x - 1 = 0$  has roots  $(2\alpha + 1), (2\beta + 1)$  and  $(2\gamma + 1)$

Substitute each root into equation e.g.

$$(2\alpha + 1)^3 - (2\alpha + 1) - 1 = 0$$

Add all the equations formed together:

$$(2\alpha + 1)^3 + (2\beta + 1)^3 + \dots$$

We get:

$$S_3 - S_1 - 3 = 0$$

We must find  $S_{-2}$  so if we multiply with  $S_{-1}$  it would result in:

$$S_2 - 3 - S_{-1} = 0$$

$$S_{-1} = S_2 - 3$$

By repeating the process (multiply with  $S_{-1}$ ) we get:

$$S_{-2} = S_1 - S_{-1}$$

By manipulating equations, we get a useful equation:

$$S_{-2} = S_1 - S_2 + 3$$

We know  $S_1 = 1$  but to find  $S_2$  use substitution

$$\text{Let } y = x^2 \quad \therefore x = \sqrt{y}$$

By substituting into the initial equation:

$$(\sqrt{y})^3 - \sqrt{y} - 1 = 0$$

$$\sqrt{y}(y) - \sqrt{y} - 1 = 0$$

$$\sqrt{y}(y - 1) = 1$$

$$y(y - 1)^2 = 1$$

$$y^3 - 2y^2 + y - 1 = 0$$

$$\text{Thus } S_2 = 2$$

$$\text{Thus } S_{-2} = 1 - 2 + 3 = 2$$

## 2. RATIONAL FUNCTIONS

### 2.1 Partial Fractions

- To split an improper fraction into partials, calculate the degree of the quotient and work out by comparing coefficients

$$\deg(Q(x)) = \deg(N(x)) - \deg(D(x))$$

- Where:

- $Q(x)$  is quotient
- $N(x)$  is numerator
- $D(x)$  is denominator

### 2.2 Vertical Asymptote

- Making denominator 0 resulting in  $\infty$

- Example:

$$y = \frac{1}{(x+1)(x-3)}$$

Thus vertical asymptotes at:  $x = -1$  and  $3$

### 2.3 Horizontal Asymptote

- By dividing the top and bottom of a fraction by  $x$ , we can see what value  $y$  tends to when  $x$  becomes very large

- Example:

$$y = \frac{3x-2}{x+1}$$

Divide numerator and denominator by  $x$

$$y = \frac{\frac{3x-2}{x}}{\frac{x+1}{x}} = \frac{3 - \frac{2}{x}}{1 + \frac{1}{x}}$$

When  $x$  is very large,  $y = \frac{3}{1} = 3$

Thus horizontal asymptote at:  $y = 3$

### 2.4 Oblique Asymptotes

- Occurs only with improper fraction

- Example:

$$y = 2x - 1 + \frac{2}{x-1} - \frac{3}{x+2}$$

When  $x$  becomes very large,  $y \approx 2x - 1$

Thus oblique asymptote at:  $y = 2x - 1$

### 2.5 Sign Tables

- Used to visualize graph as it shows in which quadrant the graph lies
- Enter values of  $x$  which result in different parts of the fraction equaling zero
- Leave columns between each value of  $x$  and place signs to indicate whether value +ve or -ve in each cell

- Example:

$$y = \frac{3x^2 + 3x + 6}{(x+3)(x-2)}$$

$x$		-3		2	
$3x^2 + 3x + 6$	+	+	+	+	+
$x + 3$	-	0	+	+	+
$x - 2$	-	-	-	0	+
$y$	+	$\infty$	-	$\infty$	+

### 2.6 Curve Sketching

- When you sketch the curve, include the following:

- $y$ -intercept
- $x$ -intercepts
- Stationary points (maxima, minima, inflections)
- Vertical asymptote(s)
- Horizontal or oblique asymptote(s)

**{S03-P01}**

**Question 4:**

The curve  $C$  has equation:

$$y = \frac{x^2 - 4}{x - 3}$$

- Find the equations of the asymptotes
- Draw a sketch of  $C$  and its asymptotes. Give the coordinates of the points of intersection of  $C$  with the coordinate axes.

**Solution:**

**Part (i):**

From the first formula:

$$\deg(Q(x)) = \deg(N(x)) - \deg(D(x))$$

$$\deg(Q(x)) = 2 - 1 = 1$$

Write the equation as partial fractions:

$$\frac{x^2 - 4}{x - 3} = Ax + B + \frac{C}{x - 3}$$

$$(Ax + B)(x - 3) + C = x^2 - 4$$

$$Ax^2 - 3Ax + Bx - 3B + C = x^2 - 4$$

Finding the coefficients using algebraic manipulation:

$$Ax^2 = x^2 \quad \therefore A = 1$$

$$-3Ax + Bx = 0x$$

$$\therefore -3A + B = 0 \quad \therefore B = 3(1) = 3$$

$$-3B + C = -4$$

$$\therefore -3(3) + C = -4 \quad \therefore C = -4 + 9 = 5$$

The final partial fraction can be expressed as:

$$y = x + 3 + \frac{5}{x - 3}$$

Vertical asymptote:

$$x - 3 = 0 \quad \therefore x = 3$$

Oblique asymptote:

$$\text{as } x \uparrow, y \approx x + 3 \quad \therefore y = x + 3$$

**Part (ii):**

We know the asymptotes but need the intercepts and stationary points

y-intercept:

$$x = 0 \quad \therefore y = 0 + 3 + \frac{5}{0-3} = \frac{4}{3}$$

$$\therefore \text{coordinates} = \left(0, \frac{4}{3}\right)$$

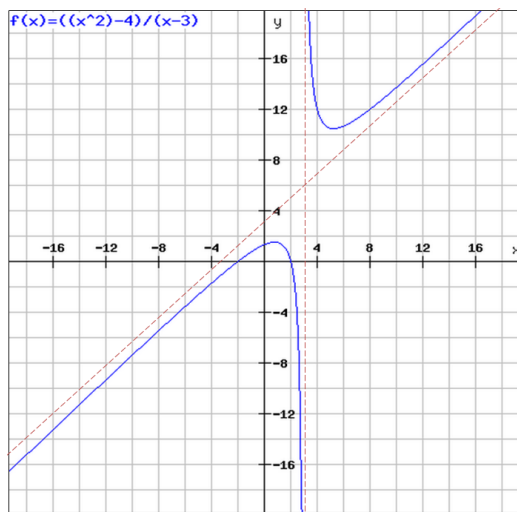
x-intercepts:

$$y = 0 \quad \therefore \frac{x^2 - 4}{x - 3} = 0$$

$$x^2 - 4 = 0 \quad \therefore x = \pm 2$$

$$\therefore \text{coordinates} = (2, 0), (-2, 0)$$

Sketch:



### 3.2 Converting between Cartesian and Polar

• Basic facts:

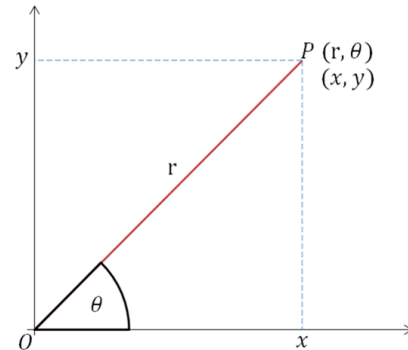
$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r^2 = x^2 + y^2$$

$$\tan \theta = \frac{y}{x}$$

• Represented on a diagram:



**{S04-P01}**

**Question 3:**

The curve  $C$  has equation

$$(x^2 + y^2)^2 = 4xy$$

Show that the polar equation of  $C$  is  $r^2 = 2 \sin 2\theta$

**Solution:**

Using identities, form an equation in terms of  $r$  and  $\theta$

$$(r^2)^2 = 4(r \cos \theta)(r \sin \theta)$$

$$r^4 = 4r^2 \cos \theta \sin \theta$$

$$r^2 = 2(2 \cos \theta \sin \theta)$$

$$r^2 = 2 \sin 2\theta$$

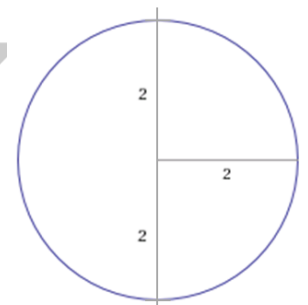
### 3.3 Sketching Polar Curves

**Circles**

•  $r = a$

○ Radius is  $a$

○ Centre of circle:  $(0, 0)$

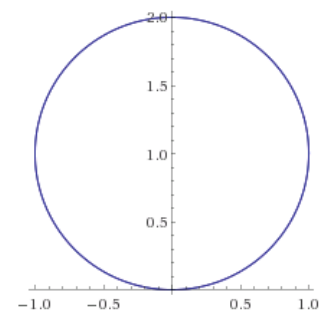


$$r = 2$$

•  $r = a \sin \theta$

○ Diameter is  $a$

○ Centre of circle:  $\left(\frac{a}{2}, \frac{\pi}{2}\right)$



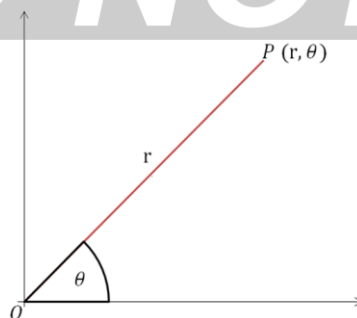
$$r = 2 \sin \theta$$

## 3. POLAR COORDINATES

### 3.1 Definitions

• A point,  $P$ , has coordinates  $(r, \theta)$  where:

- $r$  is the distance from the pole,  $O$
- $\theta$  is angle measure from base half line to radius  $OP$

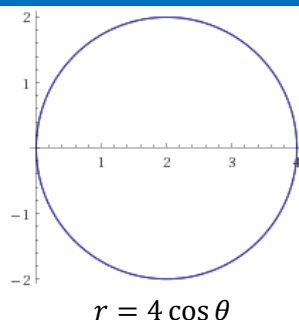


• Important points:

- $(r, \theta)$  is an ordered pair – must always be in that order
- Angle always measured positive anticlockwise, principal value which is  $-\pi < \theta < \pi$
- Angle measured in radians
- $r$  can only be positive

•  $r = a \cos \theta$

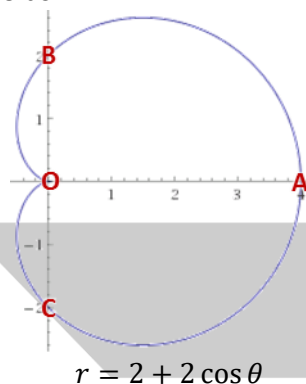
- Diameter is  $a$
- Centre of circle:  $(\frac{a}{2}, 0)$



### Cardioids

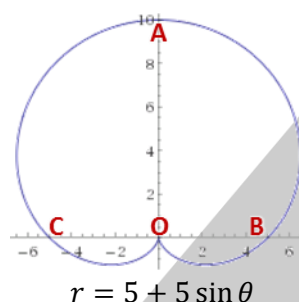
•  $r = a + a \cos \theta$

- $|OA| = 2a$
- $|OB| = |OC| = a$



•  $r = a + a \sin \theta$

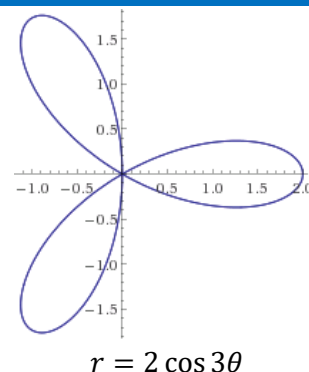
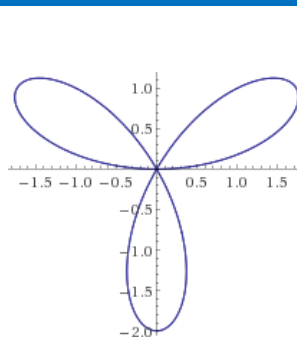
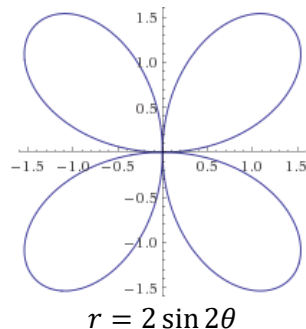
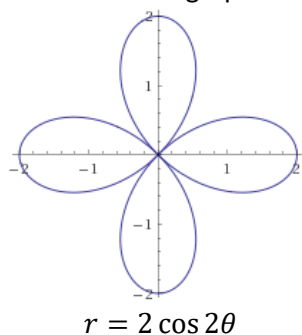
- $|OA| = 2a$
- $|OB| = |OC| = a$



### Flowers

•  $r = a \cos b\theta$  or  $a \sin b\theta$

- Length of petal =  $a$
- No. of petals:
  - if  $b$  odd then  $b$  petals
  - if  $b$  even then  $2b$  petals
- Cosine flower graphs start from  $\theta = 0^\circ$  line
- Sine flower graphs start from  $\theta = 45^\circ$  line

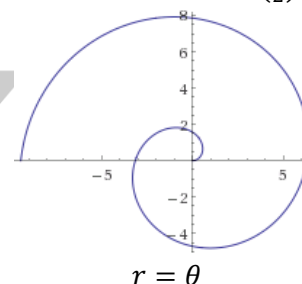


### Spirals

- When sketching spirals, first recognize the type they are, locate the centre and find intersections at  $n(\frac{\pi}{2})$

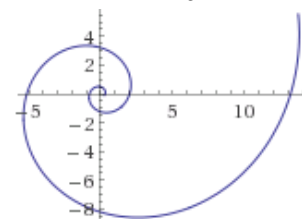
•  $r = a\theta$

- $a > 1$  looser spiral
- $a < 1$  tighter spiral
- Begins at  $(0, 0)$



•  $r = ae^{b\theta}$

- First intersection to origin =  $a$
- Begins at  $(a, 0)$



### 3.4 Extremes - Maxima, Minima & Tangents

Required Results	Maximize/Minimize	Find
Furthest point from origin	$r$	$\frac{dr}{d\theta}$
Horizontal Tangent	$y = r \sin \theta$	$\frac{dy}{d\theta}$
Vertical Tangent	$x = r \cos \theta$	$\frac{dx}{d\theta}$

### 3.5 Calculus in Polar Curves

- Area enclosed by a curve:

$$\int \frac{1}{2} r^2 \theta \cdot d\theta$$

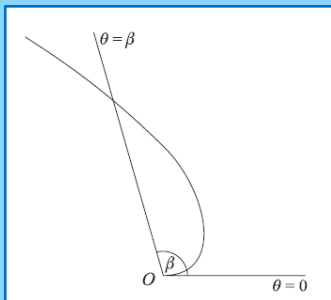
- Length of an arc:

$$\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$



{S03-P01}

Question 1:



The curve  $C$  has polar equation

$$r = \theta^{\frac{1}{2}} e^{\frac{\theta^2}{\pi}}$$

where  $0 \leq \theta \leq \pi$ . The area of the finite region bounded by  $C$  and the line  $\theta = \beta$  is  $\pi$ . Show that

$$\beta = (\pi \ln 3)^{\frac{1}{2}}$$

Solution:

Form an equation by using the area of sector formula

$$\int_0^{\beta} \frac{1}{2} r^2 \theta \cdot d\theta = A$$

$$\int_0^{\beta} \frac{1}{2} \left( \theta^{\frac{1}{2}} e^{\frac{\theta^2}{\pi}} \right)^2 \theta \cdot d\theta = \pi$$

Take  $\frac{1}{2}$  to the other side and clean up

$$\int_0^{\beta} \theta e^{\frac{2\theta^2}{\pi}} = 2\pi$$

Integrate the expression with respect to  $\theta$

$$\left[ \frac{\pi}{4} e^{\frac{2\theta^2}{\pi}} \right]_0^{\beta} = 2\pi$$

Take constant to other side and substitute  $\beta$  and 0

$$e^{\frac{2\beta^2}{\pi}} - 1 = 8$$

$$e^{\frac{2\beta^2}{\pi}} = 9$$

Take  $\ln$  on both sides and simplify

$$\frac{2\beta^2}{\pi} = \ln 9$$

$$\beta^2 = \pi \ln 3$$

$$\therefore \beta = (\pi \ln 3)^{\frac{1}{2}}$$

{W04-P01}

Question 4:

The curve  $C$  has polar equation

$$r = e^{\frac{1}{5}\theta}, \quad 0 \leq \theta \leq \frac{3}{2}\pi.$$

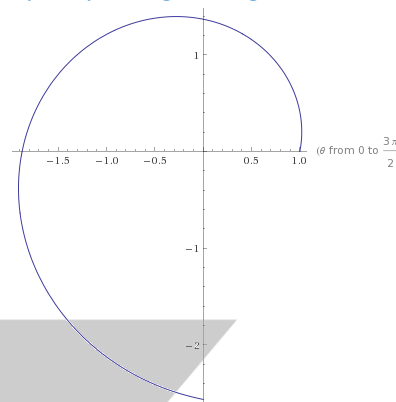
- Draw a sketch of  $C$
- Find the length of  $C$ , correct to 3 significant figures

Solution:**Part (i)**

First recognize that this is a spiral with center  $O$ . Next, work out the points of intersection at  $\frac{\pi}{2}, \pi$  and  $\frac{3\pi}{2}$

$$\left(1.37, \frac{\pi}{2}\right) \quad (1.87, \pi) \quad \left(2.57, \frac{3\pi}{2}\right)$$

Sketch the spiral passing through the intersections

**Part (ii)**

First find  $\left(\frac{dr}{d\theta}\right)^2$  and  $r^2$

$$\left(\frac{d}{d\theta} \left(e^{\frac{1}{5}\theta}\right)\right)^2 = \left(\frac{1}{5} e^{\frac{1}{5}\theta}\right)^2 = \frac{e^{\frac{2}{5}\theta}}{25}$$

$$r^2 = \left(e^{\frac{1}{5}\theta}\right)^2 = e^{\frac{2}{5}\theta}$$

Substitute into arc length formula

$$L = \int_0^{\frac{3}{2}\pi} \sqrt{e^{\frac{2}{5}\theta} + \frac{e^{\frac{2}{5}\theta}}{25}} = \int_0^{\frac{3}{2}\pi} \sqrt{\frac{26e^{\frac{2}{5}\theta}}{25}}$$

Pull out constant and integrate with respect to  $\theta$

$$= \frac{\sqrt{26}}{5} \left[ 5e^{\frac{1}{5}\theta} \right]_0^{\frac{3}{2}\pi}$$

Substitute limits and work out length

$$L = 7.99 \text{ (3s.f.)}$$

## 4. MATHEMATICAL INDUCTION

### 4.1 Proof by Induction

- Step 1: proving assertion is true for some initial value of variable
- Step 2: the inductive step
- Conclusion: final statement of what you have proved



Examples of Divisibility:

**{FP} Ex 4.1:**

**Question 2:**

Prove that  $u_n = 7^n + 4^n + 1$  is a multiple of 6

**Solution:**

**Step 1:**

Let  $n = 1$

$$\therefore u_n = 7 + 4 + 1 = 12 = 6 \times 2$$

Formula true for  $n = 1$

**Step 2:**

Assuming formula is true for  $n = k$

$$u_k = 7^k + 4^k + 1 = 6p$$

where  $p$  is just a dummy value

When  $n = k + 1$

$$u_{k+1} = 7^{k+1} + 4^{k+1} + 1$$

$$u_{k+1} = 7 \cdot 7^k + 4 \cdot 4^k + 1$$

$$u_{k+1} = 4(7^k + 4^k + 1) + 3 \cdot 7^k - 3$$

$$u_{k+1} = 4.6p + 3(7^k - 1)$$

$(7^k - 1)$  is even so can be written as  $2q$  where  $q$  is another dummy value

$$\therefore u_{k+1} = 4.6p + 3.2q$$

$$u_{k+1} = 6(4p + q)$$

If  $u_k$  is a multiple of 6, then so is  $u_{k+1}$

**Conclusion:**

By the Principle of Mathematical Induction,

$u_n = 7^n + 4^n + 1$  is a multiple of 6 for  $n \geq 1$ .

**{W02-P01}**

**Question 3:**

It is given that, for  $n = 0, 1, 2, 3, \dots$ ,

$$a_n = 17^{2n} + 3(9^n) + 20$$

Simplify  $a_{n+1} - a_n$ , and hence prove by induction that  $a_n$  is divisible by 24 for all  $n \geq 0$ .

**Solution:**

Skip Step 1 as it is already given

**Step 2:**

$$a_n = 17^{2n} + 3(9^n) + 20$$

$$a_{n+1} = 17^{2n+2} + 3(9^{n+1}) + 20$$

$$a_{n+1} = 17^2(17^{2n}) + 3 \cdot 9(9^n) + 20$$

Calculating  $a_{n+1} - a_n$ :

$$17^2(17^{2n}) + 3 \cdot 9(9^n) + 20 - 17^{2n} - 3(9^n) - 20$$

$$288(17^{2n}) + 24(9^n)$$

$$24 \cdot 12(17^{2n}) + 24(9^n)$$

$$24(12(17^{2n}) + 9^n)$$

**Conclusion:**

By the Principle of Mathematical Induction

$a_n = 17^{2n} + 3(9^n) + 20$  is a multiple of 24, for  $n \geq 0$

Example of Summation:

**{S03-P01}:**

**Question 2:**

Prove by induction that, for all  $N \geq 1$ ,

$$\sum_{n=1}^N \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(N+1)2^N}$$

**Solution:**

**Step 1:**

Let  $N = 1$ ,

$$\frac{1+2}{1(1+1)2^1} = \frac{3}{4}$$

Using the formula given:

$$1 - \frac{1}{(1+1)2^1} = 1 - \frac{1}{4} = \frac{3}{4}$$

Therefore true for  $N = 1$

**Step 2:**

Assume formula true for  $N = k$

When  $N = k$ ,

$$1 - \frac{1}{(k+1)2^k}$$

When  $N = k + 1$

$$1 - \frac{1}{(k+1+1)2^{k+1}} = 1 - \frac{1}{(k+2)2^{k+1}}$$

If formula is true then,

$$\sum_{n=1}^{k+1} \frac{n+2}{n(n+1)2^n} = \sum_{n=1}^k \frac{n+2}{n(n+1)2^n} + (k+1)^{th} \text{ term}$$

$$= 1 - \frac{1}{(k+1)2^k} + \frac{k+3}{(k+1)(k+2)2^{k+1}}$$

$$= 1 - \frac{2(k+2) - k - 3}{(k+1)(k+2)2^{k+1}}$$

$$= 1 - \frac{2k+4-k-3}{(k+1)(k+2)2^{k+1}}$$

$$= 1 - \frac{k+1}{(k+1)(k+2)2^{k+1}}$$

$$= 1 - \frac{1}{(k+2)2^{k+1}}$$

**Conclusion:**

By the Principle of Mathematical Induction, the formula is true for all  $N \geq 1$ .

## 5. SUMMATION OF SERIES

### 5.1 Standard Results of Sums

$$\sum_{r=1}^n r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$$

### 5.2 General Summation Rules

$$\sum kr = k \sum r$$

$$\sum (r+s) = \sum r + \sum s$$

$$\sum_a^b r = \sum_1^b r - \sum_1^{a-1} r$$

**{S04-P01}:**

**Question 1:**

Use the relevant standard results in the List of Formulae to prove that

$$S_N = \sum_{n=1}^N (8n^3 - 6n^2) = N(N+1)(2N^2 - 1)$$

Hence show that

$$\sum_{n=N+1}^{2N} (8n^3 - 6n^2)$$

Can be expressed in the form

$$N(aN^3 + bN^2 + cN + d)$$

Where the constants  $a, b, c, d$  are to be determined.

**Solution:**

Split up using summation rule

$$\sum_{n=1}^N (8n^3 - 6n^2) = 8 \sum_{n=1}^N n^3 - 6 \sum_{n=1}^N n^2$$

Using standard results of sums

$$= 8 \left( \frac{1}{4}n^2(n+1)^2 \right) - 6 \left( \frac{1}{6}n(n+1)(2n+1) \right)$$

$$= 2n^2(n+1)^2 - n(n+1)(2n+1)$$

$$= n(n+1)(2n(n+1) - (2n+1))$$

$$= n(n+1)(2n^2 - 1)$$

For the next part, split the summation into two parts

$$= \sum_{n=1}^{2N} (8n^3 - 6n^2) - \sum_{n=1}^{N+1-1} (8n^3 - 6n^2)$$

Using the rule above, substitute and simplify

$$= 2N(2N+1)(8N^2 - 1) - N(N+1)(2N^2 - 1)$$

$$= N((4N+2)(8N^2 - 1) - (N+1)(2N^2 - 1))$$

Expand and simplify

$$= N(30N^3 + 14N^2 - 3N - 1)$$

### 5.3 Method of Differences

**{S05-P01}:**

**Question 1:**

Use the method of differences to find  $S_N$ , where

$$S_N = \sum_{n=N}^{N^2} \frac{1}{n(n+1)}$$

Deduce the value of  $\lim_{n \rightarrow \infty} S_N$

**Solution:**

Split into separate sums

$$\sum_{n=N}^{N^2} \frac{1}{n(n+1)} = \sum_{n=1}^{N^2} \frac{1}{n(n+1)} - \sum_{n=1}^{N-1} \frac{1}{n(n+1)}$$

Split into partial fractions

$$\frac{1}{n(n+1)} \equiv \frac{1}{n} - \frac{1}{n+1}$$

Use method of differences

When added together, the following would cancel

$n = 1$	$\frac{1}{1}$	-	$\frac{1}{2}$
$n = 2$	$\frac{1}{2}$	-	$\frac{1}{3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n = N-1$	$\frac{1}{N-1}$	-	$\frac{1}{N}$
$n = N$	$\frac{1}{N}$	-	$\frac{1}{N+1}$
$\therefore \text{Sum} = 1 - \frac{1}{n+1}$			

Substitute limits

$$\sum_{n=N}^{N^2} \frac{1}{n(n+1)} = \left( 1 - \frac{1}{N^2+1} \right) - \left( 1 - \frac{1}{N} \right)$$

$$= \frac{1}{N} - \frac{1}{N^2+1}$$

Deduce the limit as the sequence converges

As  $N \rightarrow \infty$  the fractions tend to 0

$$\therefore \lim_{n \rightarrow \infty} S_N = 0$$

### 5.4 Convergence

- Finite series approaches a limit as more terms are added
- One condition is that terms must get smaller
- Satisfying this condition alone is not always sufficient
- We denote it using the following:

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N f(n) = \dots$$

### 5.5 Finding a Formula

- Specialising: trying simple examples
- Generalising: looking for a pattern
- Conjecture: suggesting a formula
- Proof: proving the formula

#### Example:

Find the  $n$ th derivative of  $xe^x$

#### Solution:

##### Step 1: Specialise

$$\begin{aligned}\frac{d}{dx}(xe^x) &= xe^x + e^x \\ \frac{d^2}{dx^2}(xe^x) &= xe^x + e^x + e^x \\ \frac{d^3}{dx^3}(xe^x) &= xe^x + e^x + e^x + e^x\end{aligned}$$

##### Step 2: Generalise

We can see that the pattern that for the  $n$ th derivative, there are  $n e^x$ s.

##### Step 3: Conjecture

$$\frac{d^n}{dx^n}(xe^x) = xe^x + ne^x$$

##### Step 4: Proof

Let  $n = k$ ,

$$\frac{d^k}{dx^k}(xe^x) = xe^x + ke^x$$

To find  $n = k + 1$ , differentiate expression:

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}(xe^x) &= xe^x + e^x + ke^x \\ &= xe^x + (k + 1)e^x\end{aligned}$$

Prove that formula gives same result,  $n = k + 1$ ,

$$\frac{d^{k+1}}{dx^{k+1}} = xe^x + (k + 1)e^x$$

By the Principle of Mathematical Induction,  $xe^x + ne^x$  is the  $n$ th derivative of  $xe^x$  for all  $n \geq 1$ .

## 6. DIFFERENTIATION

### 6.1 Second Derivative of Implicit Expressions

- Differentiate expression a second time
- Differentiating  $\frac{dy}{dx}$  gives  $\frac{d^2y}{dx^2}$

#### Example:

Differentiate  $xy^2$  twice

#### Solution:

$$\begin{aligned}y^2 + 2xy\left(\frac{dy}{dx}\right) \\ 2y\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) + 2x\left(\frac{dy}{dx}\right) \cdot \left(\frac{dy}{dx}\right) + 2xy\left(\frac{d^2y}{dx^2}\right) \\ 4y\left(\frac{dy}{dx}\right) + 2x\left(\frac{dy}{dx}\right)^2 + 2xy\left(\frac{d^2y}{dx^2}\right)\end{aligned}$$

### 6.2 Second Derivative of Implicit Equations

- May be asked to differentiate twice and use equation to find out nature of stationary points
- Nature of stationary point
  - 2<sup>nd</sup> derivative +ve = minimum
  - 2<sup>nd</sup> derivative -ve = maximum

#### Example:

Find the stationary points and their nature of the circle with equation  $x^2 - 6x + y^2 - 8y = 0$

#### Solution:

$$\begin{aligned}2x - 6 + 2y\left(\frac{dy}{dx}\right) - 8\left(\frac{dy}{dx}\right) &= 0 \\ \frac{dy}{dx} = \frac{x - 3}{y - 4}\end{aligned}$$

Stationary point: (3, -1) and (3, 9)

$$2 + 2\left(\frac{dy}{dx}\right) + 2y\left(\frac{d^2y}{dx^2}\right) - 8\left(\frac{d^2y}{dx^2}\right) = 0$$

Substitute values and find  $\frac{d^2y}{dx^2}$

$$\begin{aligned}(3, -1) \quad \frac{d^2y}{dx^2} &= \frac{1}{5} & \text{Nature} &= \text{minimum} \\ (3, 9) \quad \frac{d^2y}{dx^2} &= -\frac{1}{5} & \text{Nature} &= \text{maximum}\end{aligned}$$

### 6.3 Second Derivative of Parametric Equations

- Find first derivative simply by differentiating each parameter and dividing:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

- Find second derivative by differentiating  $\frac{dy}{dx}$  and dividing by  $\frac{dx}{dt}$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

**Example:**

Find the stationary points and their nature of the curve with parameters  $x = e^t + 1$  and  $y = 2t^2$

**Solution:**

$$\frac{dx}{dt} = e^t \quad \frac{dy}{dt} = 4t$$

$$\frac{dy}{dx} = \frac{4t}{e^t}$$

At stationary point,  $t = 0$  so  $x = 2$  and  $y = 0$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{4t}{e^t}\right)}{\frac{dx}{dt}} = \frac{\frac{4(e^t) - 4t(e^t)}{(e^t)^2}}{e^t} = \frac{4 - 4t}{e^{2t}}$$

At  $t = 0$ ,  $\frac{d^2y}{dx^2} > 0 \therefore (2, 0)$  is minimum

## 6.4 Applying Differentiation Skills

**{S03-P01}:**

**Question 7:**

The variables  $x$  and  $y$  are related by the equation  $x^4 + y^4 = 1$ , where  $0 < x < 1$  and  $0 < y < 1$ . Obtain an equation which relates  $x$ ,  $y$ ,  $\frac{dy}{dx}$ ,  $\frac{d^2y}{dx^2}$  & deduce that

$$\frac{d^2y}{dx^2} = -\frac{3x^2}{y^7}$$

**Solution:**

In order to obtain such an equation, we must differentiate twice using implicit techniques

$$4x^3 + 4y^3 \frac{dy}{dx} = 0$$

$$12x^2 + 12y^2 \left(\frac{dy}{dx}\right)^2 + 4y^3 \frac{d^2y}{dx^2} = 0$$

To obtain final equation, we must eliminate  $\frac{dy}{dx}$  from the 2<sup>nd</sup> differential hence rearranging 1<sup>st</sup> differential

$$\frac{dy}{dx} = -\frac{x^3}{y^3}$$

Rearranging second differential and substituting it in

$$\frac{d^2y}{dx^2} = \frac{-12x^2 - 12y^2 \left(-\frac{x^3}{y^3}\right)^2}{4y^3}$$

Expanding and simplifying fraction

$$\frac{d^2y}{dx^2} = \frac{-12x^2y^4 - 12x^6}{4y^7}$$

Factorizing and substituting  $y^4 = 1 - x^4$

$$\frac{d^2y}{dx^2} = \frac{-x^2(3(1 - x^4) + 3x^4)}{y^7}$$

Hence we obtain equation given

$$\frac{d^2y}{dx^2} = -\frac{3x^2}{y^7}$$

## 7. INTEGRATION

### 7.1 Reduction Formula

- Enables to solve integral problem by reducing it to an easier integral problem, and then reducing that to an easier problem, and so on.
- Generally obtained from using by integration by parts

**Example:**

Find the reduction formula for  $\int x^n e^x dx$

**Solution:**

$$I_n = \int x^n e^x dx$$

Use integration by parts formula to split up

$$u = x^n \text{ and } u' = nx^{n-1} \quad v' = e^x \text{ and } v = e^x$$

$$I_n = x^n e^x - \int nx^{n-1} e^x dx$$

You can pull out the  $n$  and place  $I_{n-1}$  instead forming

$$I_n = x^n e^x - nI_{n-1}$$

To find the integral of any value of  $n$ , you must start with  $I_0 = \int x^0 e^x - 0 dx$  hence you must first find  $\int e^x$

**Example:**

For  $I_n$  defined below, find  $I_7$

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$$

**Solution:**

Split into product to use by parts formula. Extract  $\tan^2 x$  so you can convert to  $\sec^2 x$ .

$$I_n = \int_0^{\frac{\pi}{4}} \tan^2 x \tan^{n-2} x dx$$

Use identity to switch to  $\sec^2 x$

$$I_n = \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \tan^{n-2} x dx$$

Expand brackets and split integral

$$I_n = \int_0^{\frac{\pi}{4}} \sec^2 x \tan^{n-2} x dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x dx$$

Replace final integral with  $I_{n-2}$  and split main integral using by parts rule

$$I_n = \left[ \frac{\tan^{n-1} x}{n-1} \right]_0^{\frac{\pi}{4}} - I_{n-2}$$

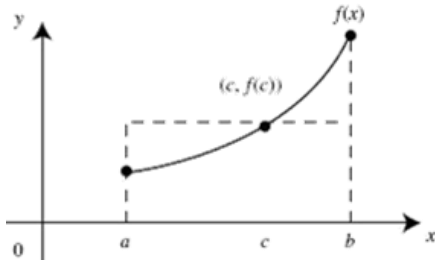
$\tan \frac{\pi}{4}$  gives 1 so first part can be simplified giving

$$I_n = \frac{1}{n-1} - I_{n-2}$$

## 7.2 Mean Value of a Function

- For a function  $f(x)$  that is continuous over  $(a, b)$  then the average value of the function is defined as

$$\bar{y} = \frac{1}{b-a} \int_a^b f(x) dx$$



- The area of the rectangle shown in dotted lines is equal to the area under the curve of  $f(x)$  over  $(a, b)$  hence we have effectively found the mean height of the curve

**{S08-P01}:**

**Question 6:**

The curve  $C$  is defined parametrically by

$$x = 4t - t^2 \quad \text{and} \quad y = 1 - e^{-t}$$

Also, you are given

$$\frac{d^2y}{dx^2} = \frac{(t-1)e^{-t}}{4(2-t)^3}$$

Show that the mean value of  $\frac{d^2y}{dx^2}$  with respect to  $x$  over the interval  $0 \leq x \leq 7$  is

$$\frac{4e^{-\frac{1}{2}} - 3}{21}$$

**Solution:**

Using the formula above

$$\bar{y} = \frac{1}{7-0} \int_0^7 \frac{(t-1)e^{-t}}{4(2-t)^3} dx$$

As we are integrating  $\frac{d^2y}{dx^2}$ , it is logical to say that its integral is  $\frac{dy}{dx}$  hence we can work that out from original parameters.

$$\frac{dx}{dt} = 4 - 2t \quad \frac{dy}{dt} =$$

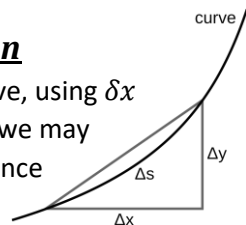
However we must change the limits from  $x$  to  $t$  using the  $x$  parameter

Hence interval:  $0 \leq t$

## 7.3 Arc Length of a Function

- When finding the length of a curve, using  $\delta x$  is not accurate enough however we may consider  $\delta s$  to be straight and hence using Pythagoras' Theorem:

$$\delta s^2 = \delta x^2 + \delta y^2$$



- We can then integrate to obtain  $s$  however before this, we must first factorize to obtain an equation we can work with

**For cartesian curves:**

- Factorizing  $\delta x$

$$\delta s^2 = \left(1 + \frac{\delta y^2}{\delta x^2}\right) \delta x^2$$

Square rooting and integrating this:

$$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

(can factorize  $\delta y$  instead and integrate with respect to  $y$ )

**For cartesian curves defined parametrically:**

- Factorizing  $\delta t$

$$\delta s^2 = \left(\frac{\delta x^2}{\delta t^2} + \frac{\delta y^2}{\delta t^2}\right) \delta t^2$$

Square rooting and integrating this:

$$s = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

**For polar curves (refer to 3.5):**

- Factorizing  $\delta \theta$

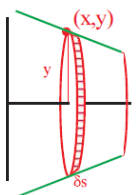
$$\delta s^2 = \left(r^2 + \frac{\delta r^2}{\delta \theta^2}\right) \delta \theta^2$$

Square rooting and integrating this:

$$s = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

## 7.4 Surface Area of a Solid Revolution

- When calculating surface area, we split curve into discs similar to when we find volume however use  $\delta s$  instead of  $\delta x$  for greater accuracy, treating the disc as the frustum of a cone



$$A = \int 2\pi y \delta s$$

- Using our equation formed for  $\delta s$ , substitute into expression and integrate

**For cartesian curves:**

- Factorizing  $\delta x$

$$\delta s^2 = \left(1 + \frac{\delta y^2}{\delta x^2}\right) \delta x^2$$

Square rooting and adding this to surface area equation:

$$A = \int 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

**For cartesian curves defined parametrically:**

- Factorizing  $\delta t$

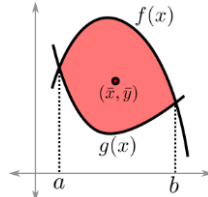
$$\delta s^2 = \left( \frac{\delta x^2}{\delta t^2} + \frac{\delta y^2}{\delta t^2} \right) \delta t^2$$

Square rooting and adding this to surface area equation:

$$A = \int 2\pi y \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} dt$$

### 7.5 Centroid of a 2-D Shape

- Can find the centroid (center of mass) by dividing the total moment of each point mass by total mass hence;



$$\bar{x} = \frac{\text{Total Moment about } y}{\text{Total Mass}} = \frac{M_y}{m}$$

$$\bar{y} = \frac{\text{Total Moment about } x}{\text{Total Mass}} = \frac{M_x}{m}$$

- Considering the curve in strips and adding density, the following equations can be formed for the different components:

**Mass:**  $m = \int \rho y \, dx$

**Moment about x:**  $M_x = \int \rho xy \, dx$

**Moment about y:**  $M_y = \int \frac{1}{2} \rho y^2 \, dx$

- For general cartesian equations, substitute directly into above equations and find the coordinates of the centroid
- For cartesian equations defined in parameters, you must first change the  $dx$  to  $dt$  in order to integrate with respect to  $t$  so:
  - Differentiate parameter of  $x$
  - Rearrange differential to make it  $dx = \dots dt$
  - Substitute into above equations and integrate

### 7.6 Centroid of a 3-D Shape

- Same concept can be applied to centroids however this time we use volumes of the solid generated
- Volumes are generated around an axis therefore the  $x$ -axis is a line of symmetry so we know  $\bar{y} = 0$

$$\bar{x} = \frac{\text{Total Moment about } y}{\text{Total Mass}} = \frac{M_y}{m}$$

- Considering the curve in discs and adding density, the following equations can be formed for the different components (moment about  $x$  not needed):

**Mass:**  $m = \int \rho \pi y^2 \, dx$

**Moment about y:**  $M_y = \int \rho \pi x y^2 \, dx$

- Follow same procedure as that for 2D shapes to find centroid of curves defined parametrically:
  - Differentiate parameter of  $x$
  - Rearrange differential to make it  $dx = \dots dt$
  - Substitute into above equations and integrate

**{S05-P01}:**

**Question 5:**

The curve  $C$  is defined parametrically by

$$x = t - 8t^{\frac{1}{2}} \quad \text{and} \quad y = \frac{16}{3} t^{\frac{3}{4}}$$

The arc of this curve joining the point where  $t = 1$  to the point where  $t = 4$  is denoted by  $C$ .

- Show that the length of  $C$  is 11.
- Find, correct to 3 significant figures, the area of the surface generated when  $C$  is rotated through 1 complete revolution about  $x$ -axis.

**Solution:**

**Part (i):**

Find the derivatives of given variables:

$$\frac{\delta x}{\delta t} = 1 - 4t^{-\frac{1}{2}} \quad \text{and} \quad \frac{\delta y}{\delta t} = 4t^{-\frac{1}{4}}$$

Variables are defined parametrically so we use the following equation:

$$\begin{aligned} s &= \int \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} dt \\ s &= \int_1^4 \sqrt{\left( 1 - 4t^{-\frac{1}{2}} \right)^2 + \left( 4t^{-\frac{1}{4}} \right)^2} dt \\ s &= \int_1^4 \sqrt{1 + 8t^{-\frac{1}{2}} + 16t^{-1}} dt \end{aligned}$$

This can be rearranged to form a complete square:

$$\begin{aligned} s &= \int_1^4 \sqrt{16t^{-1} + 4t^{-\frac{1}{2}} + 4t^{-\frac{1}{2}} + 1} dt \\ s &= \int_1^4 \sqrt{\left( 4t^{-\frac{1}{2}} + 1 \right)^2} dt \\ s &= \int_1^4 \left( 4t^{-\frac{1}{2}} + 1 \right) dt \\ s &= \left[ 8t^{\frac{1}{2}} + t \right]_1^4 = 11 \end{aligned}$$



**Part (ii):**

To find area about  $x$ -axis with parametric equations use the following:

$$A = \int 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Substitute known variables:

$$A = 2\pi \int_1^4 \left(\frac{16}{3}t^{\frac{3}{4}}\right) \left(4t^{-\frac{1}{2}} + 1\right) dt$$

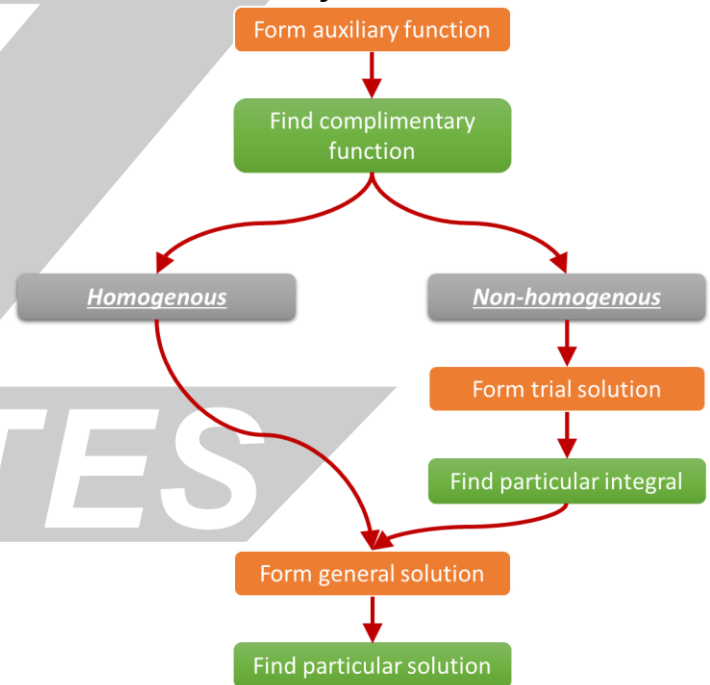
$$A = 2\pi \int_1^4 \frac{64}{3}t^{\frac{1}{4}} + \frac{16}{3}t^{\frac{3}{4}} dt$$

$$A = 2\pi \left[ \frac{256}{15}t^{\frac{5}{4}} + \frac{64}{21}t^{\frac{7}{4}} \right] = 697 \text{ (3s.f.)}$$

- After finding a complementary function we have to add a particular integral
- The particular integral we add is dependent upon  $f(x)$  and the trial solution used:

$f(x)$	Trial Solution
$ax^2 + bx + c$	$Px^2 + Qx + R$
$a \sin bx / a \cos bx$	$P \cos bx + Q \sin bx$
$ae^{bx}$	$Pe^{bx}$

- Once we have the trial solution we equate it to  $y$  and differentiate it to find  $y'$  and differentiate again to find  $y''$ , in terms of  $P$ ,  $Q$  and  $R$
- Then substitute these into the initial differential equation and solve simultaneously for  $P$ ,  $Q$  and  $R$
- The trial solution with substituted values of  $P$ ,  $Q$  and  $R$  becomes the particular integral
- Addition of complementary function and particular integral gives us the particular solution

**8.3 Method Summary****8. DIFFERENTIAL EQUATIONS**

- The type of differential equations we solve are linear, second order, constant coefficient differential equations
- There are two types:
  - Homogenous
  - Non-Homogenous

**8.1 Homogenous Equations**

- These type of differential equations will be of the form:

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

- First step to solving them is to find the complementary function by forming an auxiliary equation:

$$a\lambda^2 + b\lambda + c = 0$$

- The complementary function formed depends on the discriminant of the auxiliary function
- Solve the quadratic equation and use the following cases to work out complementary function

Discriminant	Solutions	Complementary Function
$b^2 - 4ac > 0$	$m, n$	$Ae^{mx} + Be^{nx}$
$b^2 - 4ac = 0$	$m, m$	$(Ax + B)e^{mx}$
$b^2 - 4ac < 0$	$p + iq, p - iq$	$e^{px}(A \cos qx + B \sin qx)$

- The complementary function is also the general solution and we need to substitute values, find  $A$  and  $B$  to get the particular solution

**8.2 Non-Homogenous Equations**

- These type of differential equations will be of the form:

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$$

- We find a complementary function in the same way as above, by making  $f(x) = 0$

**{W02-P01}:****Question 8:**

The value of the assets of a large commercial organization at time  $t$ , measured in years, is  $\$(10^8y + 10^9)$ . The variables  $y$  and  $t$  are related by the differential equation:

$$\frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y = 15 \cos 3t - 3 \sin 3t$$

Find  $y$  in terms of  $t$ , given that  $y = 3$  and  $\frac{dy}{dt} = 2$  when  $t = 0$ .



**Solution:**

Form the auxiliary equation by assuming  $f(x) = 0$ :

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

$$\therefore \lambda^2 + 5\lambda + 6 = 0$$

Solve the quadratic equation

$$(\lambda + 3)(\lambda + 2) = 0$$

$$\therefore \lambda = 3 \text{ or } 2$$

Answers are two real solutions so the complementary function is the following:

$$y = Ae^{-3t} + Be^{-2t}$$

Now we have to find the particular integral using a trial solution:

$$f(x) = 15 \cos 3t - 3 \sin 3t$$

Thus our trial solution and its derivatives are:

$$y = P \cos 3t + Q \sin 3t$$

$$y' = -3P \sin 3t + 3Q \cos 3t$$

$$y'' = -9P \cos 3t - 9Q \sin 3t$$

Substitute into original equation and simplify to get:

$$(-3P + 15Q) \cos 3t + (-3Q + 15P) \sin 3t$$

Take the coefficients from both sides to form simultaneous equations:

$$-3P + 15Q = 15$$

$$-3Q + 15P = -3$$

Solve these to get  $P$  and  $Q$

$$P = 0 \text{ and } Q = 1$$

Thus the particular integral is:

$$y = \sin 3t$$

Our general solution is the addition of the complementary function and the particular integral:

$$y = Ae^{-3t} + Be^{-2t} + \sin 3t$$

First step to finding particular solution is to differentiate general solution:

$$\frac{dy}{dt} = -3Ae^{-3t} - 2Be^{-2t} + \cos 3t$$

Substitute given information into the equations above to form simultaneous equations

$$\text{From } y: 3 = A + B$$

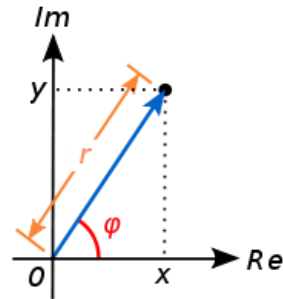
$$\text{From } \frac{dy}{dt}: 3A + 2B = -1$$

Solve simultaneous equations to find  $A$  and  $B$

$$A = -7 \text{ and } B = 10$$

Thus our particular solution is:

$$y = -7e^{-3t} + 10e^{-2t} + \sin 3t$$

**9. COMPLEX NUMBERS****9.1 Representing Complex Numbers**

<b>Cartesian Format</b>	$x + iy$	$(x, y)$
<b>Polar Format</b>	$r(\cos \theta + i \sin \theta)$	$(r, \theta)$
<b>Exponential Format</b>	$re^{i\theta}$	$(r, \theta)$

**9.2 De Moivre's Theorem**

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

- If the polar coordinates of  $z$  are  $(r, \theta)$
- Then the polar coordinates of  $z^n$  are  $(r^n, n\theta)$

**9.3 Finding Roots****Example:**

Solve the following equation, giving your answer in exponential format and plotting the results on an Argand diagram

$$z^5 + 16\sqrt{3} - 16i = 0$$

**Solution:**

$$z^5 = -16\sqrt{3} + 16i$$

Firstly, find the modulus and argument of  $z^5$

$$r = \sqrt{(-16\sqrt{3})^2 + 16^2} = 32$$

Sketch to find what angle you are looking for

$$\theta = 180 - \tan^{-1} \frac{16}{16\sqrt{3}} = 120^\circ$$

Using De Moivre's Theorem, find the modulus and argument of  $z$

$$r = \sqrt[5]{32} = 2$$

$$\theta = \frac{120 + 360n}{5}$$

Substitute values of  $n$  from 0 to the power i.e. 5  
Must convert angles out of the range to bring between  $-180$  to  $180$ . Do this by sketching the angle quickly. Then convert to radians.

	Angle	Argument	Radians
$n = 0$	$\theta = 24^\circ$	$\theta = 24^\circ$	$\frac{2}{15}\pi$
$n = 1$	$\theta = 96^\circ$	$\theta = 96^\circ$	$\frac{8}{15}\pi$
$n = 2$	$\theta = 168^\circ$	$\theta = 168^\circ$	$\frac{14}{15}\pi$
$n = 3$	$\theta = 240^\circ$	$\theta = -120^\circ$	$-\frac{10}{15}\pi$
$n = 4$	$\theta = 312^\circ$	$\theta = -48^\circ$	$-\frac{4}{15}\pi$
$n = 5$	$\theta = 384^\circ$	$\theta = 24^\circ$	$\frac{2}{15}\pi$

Therefore the roots have:

Modulus = 2 and arguments as above

Express in exponential format

$$z = e^{in\frac{2}{15}\pi} \text{ where } n = 1, 4, 7, -5, -2$$

## 9.4 Binomial Expansions

- Using De Moivre's and the binomial theorems we can express  $\cos n\theta$  and  $\sin n\theta$  in terms of  $\cos \theta$  and  $\sin \theta$

**Example:**

Find expressions for  $\cos 5\theta$  and  $\sin 5\theta$  in terms of  $\cos \theta$  and  $\sin \theta$ .

**Solution:**

Using De Moivre's Theorem

$$\cos 5\theta + i \sin 5\theta = (\cos \theta + i \sin \theta)^5$$

Expand the right-hand side using binomial theorem:

$$\cos^5 \theta + 5i \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^2 \theta - 10i \cos^2 \theta \sin^3 \theta + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta$$

Equate real variables:

$$\begin{aligned} \cos 5\theta &= \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta \\ &= 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta \end{aligned}$$

Equate imaginary variables:

$$\begin{aligned} \sin 5\theta &= \sin^5 \theta - 10 \cos^2 \theta \sin^3 \theta + 5 \cos^4 \theta \sin \theta \\ &= 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta \end{aligned}$$

## 9.5 Trigonometric Identities

- We know that:

$$\begin{aligned} z &= e^{i\theta} = \cos \theta + i \sin \theta \\ z^* &= e^{-i\theta} = \cos \theta - i \sin \theta \end{aligned}$$

- From these we can formulate trigonometric identities:

$$\begin{aligned} \cos \theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) & \sin \theta &= \frac{1}{2}(e^{i\theta} - e^{-i\theta}) \\ \cos n\theta &= \frac{1}{2}(e^{in\theta} + e^{-in\theta}) & \sin n\theta &= \frac{1}{2}(e^{in\theta} - e^{-in\theta}) \end{aligned}$$

**Example:**

Find expressions for  $\cos^5 \theta$  in terms of cosines of multiple angles.

**Solution:**

$$\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\therefore \cos^5 \theta = \frac{1}{32}(e^{i\theta} + e^{-i\theta})^5$$

Expand using the binomial theorem:

$$\begin{aligned} \cos^5 \theta &= \frac{1}{32}(e^{5i\theta} + 5e^{i3\theta} + 10e^{i\theta} + 10e^{-i\theta} \\ &\quad + 5e^{-i3\theta} + e^{-5i\theta}) \end{aligned}$$

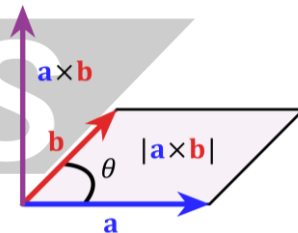
Rearrange this:

$$\begin{aligned} \cos^5 \theta &= \frac{1}{16} \left( \frac{1}{2}(e^{i5\theta} + e^{-i5\theta}) + \frac{5}{2}(e^{i3\theta} + e^{-i3\theta}) \right. \\ &\quad \left. + \frac{10}{2}(e^{i\theta} + e^{-i\theta}) \right) \end{aligned}$$

$$\cos^5 \theta = \frac{1}{16}(\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta)$$

## 10. VECTORS

### 10.1 Vector Product



- The vector product of two vectors results in the common perpendicular to both vectors
- For two vectors  $\mathbf{a}$  and  $\mathbf{b}$ :

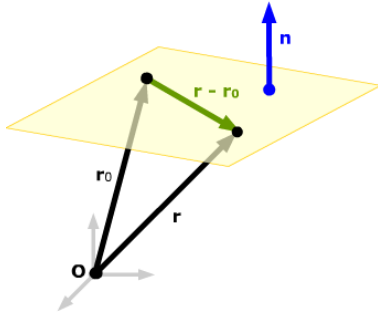
$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

The vector product can be found the determinant of a matrix consisting of the two vectors:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- You can calculate the angle between the two vectors by  $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$

### 10.2 Equation of a Plane



- **Parametric form:** a plane is made up of two direction vectors hence can be written as

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}_1 + \mu \mathbf{d}_2$$

- **Scalar product form:** find the normal vector by finding cross product of the two direction vectors. Find  $D$  by substituting a point in  $\mathbf{r}$

$$\tilde{\mathbf{r}} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = D$$

- **Cartesian form:** coefficients are components of the normal vector

$$n_1x + n_2y + n_3z = D$$

### 10.3 Finding the Equation of a Plane

- **Given 3 points on a plane:**

$$A = (1, 2, -1), B = (2, 1, 0), C = (-1, 3, 2)$$

- Find 2 direction vectors e.g.  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  (can be any pair) and find the cross product. This is the normal:

$$\therefore \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -4 \\ -5 \\ -1 \end{pmatrix}$$

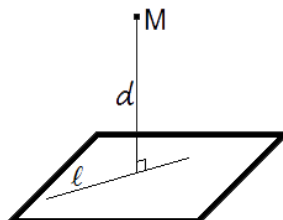
- Substitute point  $A$  to get  $D$

$$\therefore \mathbf{r} \cdot \begin{pmatrix} -4 \\ -5 \\ -1 \end{pmatrix} = -13$$

- **Given a point and a line on the plane:** Make 2 points on the line by substituting different values for  $\lambda$ . Repeat the 3 point process as above.
- **Given 2 lines on a plane:** Find a point on one line and 2 points on the other line by substituting different values for  $\lambda$ . Repeat the 3 point process as above.

### 10.4 $\perp$ Distance from a Point to a Plane

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$



### 10.5 Line of Intersection of Two Planes

- The direction of the line of intersection would be normal to both the normal of the planes so

$$\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2$$

- To find a point on the plane, set one of the variables to a value and solve to find two other points
- *Two ways there is no line of intersection:*
  - Planes may be parallel – if so normal vectors would be the same (or negative)
  - May be the same plane with different equations

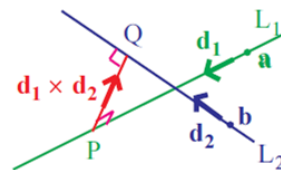
### 10.6 Intersection of a Line and a Plane

- Find point by substituting line as  $\mathbf{r}$  into the scalar product of the plane, find  $\lambda$  and find coordinates

- *Two ways there is no point of intersection:*

- Line is parallel to the plane – equation with  $\lambda$ s won't solve and  $\mathbf{d} \cdot \mathbf{n} = 0$
- Line lies in the plane – equation ends with  $0 = 0$  and any point on the line will be a solution. Also  $\mathbf{d} \cdot \mathbf{n} = 0$

### 10.7 Distance between Two Skew Lines



$$L_1: \mathbf{r} = \mathbf{a} + \lambda \mathbf{d}_1$$

$$L_2: \mathbf{r} = \mathbf{b} + \mu \mathbf{d}_2$$

- Observing diagram above, one can follow line  $L_1$  to point  $P$  and then moving along the normal of the two lines to point  $Q$ . This can be represented by a line as

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}_1 + t(\mathbf{d}_1 \times \mathbf{d}_2)$$

- This point can also be reached simply with line  $L_2$ . Hence as they both get to the same point, we can equate above line and  $L_2$

$$\mathbf{a} + \lambda \mathbf{d}_1 + t(\mathbf{d}_1 \times \mathbf{d}_2) = \mathbf{b} + \mu \mathbf{d}_2$$

- Form three equations using each coordinate and solve to find  $\lambda$ ,  $t$  and  $\mu$ .
- The perpendicular distance required between the two skew lines is  $|t(\mathbf{d}_1 \times \mathbf{d}_2)|$

### Equation of the Line of the Shortest Distance:

- The equation we are looking for is  $PQ$ , the one we formed initially:

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}_1 + t(\mathbf{d}_1 \times \mathbf{d}_2)$$

- Substitute all the values, simplify, and form the equation with the parameter  $t$

## 10.8 Angles

$$\frac{a \cdot b}{|a||b|} = \cos \theta$$

- **Angle between two skew lines:**
  - Dot product between the two direction vectors
  - If vectors in opposite directions, find obtuse angle
- **Angle between line and plane:**
  - Dot product between the line's direction vector and the plane's normal
  - Angle found is with the normal so do  $90 - \theta$
- **Angle between line and plane:**
  - Dot product between their normals
  - If obtuse find equivalent acute
- When using dot product rule to find an angle,

Question asks  
for acute angle

Use +ve value  
of dot product

Question asks  
for obtuse angle

Use -ve value  
of dot product

Question asks  
for both angles

Use +ve and -  
ve value of dot  
product

## 11. MATRICES

### 11.1 Eigenvalues, Eigenlines and Eigenvectors of $2 \times 2$ Matrices

- **Eigenvalues:** the scale factors of the transformation produced by a matrix
- **Eigenlines:** invariant lines through the origin; a line where objects on it are not changed by the matrix
- **Eigenvectors:** any vector parallel to an eigenline; vectors that do not change direction when matrix applied

#### Example:

Find the eigenvalues, eigenlines and eigenvectors for the following matrix

$$\begin{pmatrix} 10 & -6 \\ 12 & -7 \end{pmatrix}$$

#### Solution:

Using the characteristic equation

$$\text{Det}(\mathbf{M} - \lambda \mathbf{I}) = 0$$

$$\text{Det} \left( \begin{pmatrix} 10 & -6 \\ 12 & -7 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \begin{vmatrix} 10 - \lambda & -6 \\ 12 & -7 - \lambda \end{vmatrix}$$

Form an equation and solve to find eigenvalues

$$(10 - \lambda)(-7 - \lambda) - (-6)(12) = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda = 2 \text{ or } 1$$

Next, find the eigenvectors by multiplying by original matrix and solving equation by putting values

$$\lambda = 2 \quad \begin{pmatrix} 10 & -6 \\ 12 & -7 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix}$$

$$10a - 6b = 2a$$

$$8a = 6b$$

$$a = 3 \quad b = 4$$

$$\lambda = 1 \quad \begin{pmatrix} 10 & -6 \\ 12 & -7 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$10a - 6b = a$$

$$9a = 6b$$

$$a = 2 \quad b = 3$$

Hence, we have found

$$\lambda = 2, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ and } y = \frac{4}{3}x$$

$$\lambda = 1, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } y = \frac{3}{2}x$$

### 11.2 Matrix Decomposition

- Matrix transformation made up of following component:
  - **Matrix  $\mathbf{Q}^{-1}$ :** Transform the eigenlines so that they lie along the  $x$  and  $y$  axes
  - **Matrix  $\mathbf{D}$ :** Dilate along the axes using the eigenvalues as scale factors
  - **Matrix  $\mathbf{Q}$ :** Transform axes back to original eigenlines
- Finding matrix  $\mathbf{Q}$ : place two eigenvectors side-by-side

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

- Finding matrix  $\mathbf{D}$ : place scale factors in the corners

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Note: the eigenvalue should be in the same column as the corresponding eigenvector

#### Example:

Decompose the following matrix into the form  $\mathbf{QDQ}^{-1}$

$$\begin{pmatrix} 10 & -6 \\ 12 & -7 \end{pmatrix}$$

#### Solution:

From the previous part:

$$\lambda = 2, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ and } \lambda = 1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Using method from above,

$$\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \mathbf{Q} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$$

To find  $\mathbf{Q}^{-1}$ , find the inverse of  $\mathbf{Q}$

$$\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix} = 9 - 8 = 1$$

$$\mathbf{Q}^{-1} = \frac{1}{1} \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} 10 & -6 \\ 12 & -7 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$$

### 11.3 Powers of Matrices

- Decomposition of matrices provides a convenient way of calculating powers of a matrix

$$\mathbf{M}^n = \mathbf{Q} \times \mathbf{D}^n \times \mathbf{Q}^{-1}$$

**Example:**

Find  $\mathbf{M}^5$

$$\mathbf{M} = \begin{pmatrix} 10 & -6 \\ 12 & -7 \end{pmatrix}$$

**Solution:**

From the previous part:

$$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$$

Applying power to the  $\mathbf{D}$  matrix

$$\begin{aligned} \mathbf{M}^5 &= \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^5 \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 32 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix} \end{aligned}$$

Multiply, starting with the last two matrices

$$\begin{pmatrix} 32 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 96 & -64 \\ -4 & 3 \end{pmatrix}$$

Multiply this matrix with the first matrix

$$\begin{aligned} \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \times \begin{pmatrix} 96 & -64 \\ -4 & 3 \end{pmatrix} &= \begin{pmatrix} 280 & -186 \\ 372 & -247 \end{pmatrix} \\ \therefore \mathbf{M}^5 &= \begin{pmatrix} 280 & -186 \\ 372 & -247 \end{pmatrix} \end{aligned}$$

### 11.4 Determinant of a Square Matrix

- We must use laplace expansion to find the matrix of a 3 by 3 matrix
- When choosing a row to expand with, you must apply a sign to it. The top corner is +ve after which everything is alternating:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

**Example:**

Find the determinant of

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

**Solution:**

Laplace expansion using the first row and applying the signs:

$$\begin{aligned} &\text{expanding with first row} \quad \begin{matrix} + & - & + \\ \text{①} & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix} \quad \begin{matrix} - & + & - \\ 1 & \text{②} & 3 \\ 4 & 5 & \text{⑥} \\ 7 & 8 & 9 \end{matrix} \quad \begin{matrix} + & - & + \\ 1 & 2 & \text{③} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix} \\ &= 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \end{aligned}$$

Find the determinant of each matrix

$$[1 \times (45 - 48)] - [2 \times (36 - 42)] + [3 \times (32 - 35)]$$

Hence find the determinant of the 3 by 3 matrix

$$(-3) - (-12) + (-9) = 0$$

### 11.5 Inverse of a 3-by-3 Matrix

Using minors and co-factors:

- Find the determinant of the matrix
- Find the determinant of the minor of every element in the matrix and form a matrix with these determinants
- Add + and - sign to each element
- Transpose the matrix
- Divide matrix by the determinant of the original matrix

**Example:**

Find the inverse of

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

**Solution:**

Find the determinant of the matrix

$$\begin{aligned} &= 1 \begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ -2 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} \\ &= (-2) - (2) + (2) = -2 \end{aligned}$$

Find the minors of the matrix. To find a minor e.g. of the element (1, 1), cover the first row and column and select the 2 by 2 matrix left.

1<sup>st</sup> Column

1<sup>st</sup> Row

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

Find each co-factor and form a matrix with the determinant of each as an element

$$\begin{pmatrix} \begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ -2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \end{pmatrix}$$

Find each determinant

$$\begin{pmatrix} -2 & 2 & -2 \\ 2 & 0 & 2 \\ -1 & 1 & -2 \end{pmatrix}$$

Transpose the matrix, switch the element at the end of each diagonal

Divide by original determinant and therefore the inverse of the matrix is

$$\mathbf{M}^{-1} = \frac{1}{-2} \begin{pmatrix} -2 & 2 & -1 \\ 2 & 0 & 1 \\ -2 & 2 & -2 \end{pmatrix}$$

## 11.6 Eigenvalues and Eigenvectors of $3 \times 3$ Matrices

**Example:**

Find the eigenvalues and eigenvectors for the following matrix

$$\begin{pmatrix} 9 & 12 & 13 \\ 5 & 2 & 5 \\ -12 & -12 & -16 \end{pmatrix}$$

**Solution:**

Using the characteristic equation

$$\begin{aligned} \text{Det}(\mathbf{M} - \lambda \mathbf{I}) &= 0 \\ \text{Det} \left( \begin{pmatrix} 9 & 12 & 13 \\ 5 & 2 & 5 \\ -12 & -12 & -16 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \\ &= \begin{vmatrix} 9-\lambda & 12 & 13 \\ 5 & 2-\lambda & 5 \\ -12 & -12 & -16-\lambda \end{vmatrix} \end{aligned}$$

Find determinant using laplace expansion:

$$\begin{aligned} (9-\lambda) \begin{vmatrix} 2-\lambda & 5 \\ -12 & -16-\lambda \end{vmatrix} &= \\ (9-\lambda)(2-\lambda)(-16-\lambda) + 60(9-\lambda) & \\ -12 \begin{vmatrix} 5 & 5 \\ -12 & -16-\lambda \end{vmatrix} &= -60(-16-\lambda) + 720 \end{aligned}$$

$$3 \begin{vmatrix} 5 & 2-\lambda \\ -12 & -12 \end{vmatrix} = -180 + 36(2-\lambda)$$

Merging the expansions and equation to 0

$$(9-\lambda)(2-\lambda)(-16-\lambda) + 60(9-\lambda) - 60(-16-\lambda) + 720 - 180 + 36(2-\lambda) = 0$$

Expand, simplify and factorize the cubic and solve hence finding the eigenvalues

$$\begin{aligned} (\lambda + 4)(\lambda + 3)(\lambda - 2) &= 0 \\ \lambda &= -4, -3, 2 \end{aligned}$$

Solve, as 2-by-2 matrices, the eigenvectors

$$\lambda = -4 \quad \begin{pmatrix} 9 & 12 & 13 \\ 5 & 2 & 5 \\ -12 & -12 & -16 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4x \\ -4y \\ -4z \end{pmatrix}$$

$$\lambda = -3 \quad \begin{pmatrix} 9 & 12 & 13 \\ 5 & 2 & 5 \\ -12 & -12 & -16 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3x \\ -3y \\ -3z \end{pmatrix}$$

$$\lambda = 2 \quad \begin{pmatrix} 9 & 12 & 13 \\ 5 & 2 & 5 \\ -12 & -12 & -16 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Multiply each matrix, equating to the right to form equations and solve the three simultaneous equation.

Assuming all steps done, vectors found are:

$$\lambda = -4, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \lambda = -3, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \lambda = 2, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

## 12. VECTOR SPACES

### 12.1 Definition

- **Vector Space:** a structure that consists of a set of vectors,  $\mathbf{V}$ , and a set of scalars,  $\mathbf{S}$ , which must satisfy all axioms

- E.g. A vector space defined as follows:

$$\mathbf{V} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \quad \mathbf{S} = \{S_1, S_2, S_3\}$$

$$\mathbf{a} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$\mathbf{b} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$$

$$\mathbf{c} = g\mathbf{i} + h\mathbf{j} + i\mathbf{k}$$

- The set of vectors in matrix form:

$$\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

- The vector space in equation form::

$$S_1 \begin{pmatrix} a \\ b \\ c \end{pmatrix} + S_2 \begin{pmatrix} d \\ e \\ f \end{pmatrix} + S_3 \begin{pmatrix} g \\ h \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- **Axiom:** a rule that the elements of the vector space must follow; there are two types - addition and multiplication

### 12.2 Addition Axioms

- **A1 – Closure:**

$$\mathbf{a} \oplus \mathbf{b} \in \mathbf{V}$$

- **A2 – Associativity:**

$$(\mathbf{a} \oplus \mathbf{b}) \oplus \mathbf{c} = \mathbf{a} \oplus (\mathbf{b} \oplus \mathbf{c})$$

- **A3 – Identity:**

$$\mathbf{a} \oplus \mathbf{z} = \mathbf{z} \oplus \mathbf{a} = \mathbf{a}, \quad \text{for all } \mathbf{a} \in \mathbf{V}$$

- **A4 – Additive Inverse:**

$$\mathbf{a} \oplus \bar{\mathbf{a}} = \bar{\mathbf{a}} \oplus \mathbf{a} = \mathbf{z}$$

- **A5 – Commutativity:**

$$\mathbf{a} \oplus \mathbf{b} = \mathbf{b} \oplus \mathbf{a}$$

### 12.3 Multiplication Axioms

- **M1 – Closure:**

$$\lambda \mathbf{a} \in \mathbf{V}$$

- **M2 – Associativity:**

$$\lambda(\mu \mathbf{a}) = (\lambda\mu) \mathbf{a}$$

- **M3 – Identity:**

$$1\mathbf{a} = \mathbf{a}$$

- **M4 – Distributivity 1:**

$$\lambda(\mathbf{a} \oplus \mathbf{b}) = \lambda \mathbf{a} + \lambda \mathbf{b}$$

- **M5 – Distributivity 2:**

$$(\lambda + \mu) \mathbf{a} = \lambda \mathbf{a} \oplus \mu \mathbf{a}$$



### 12.4 Linear Independence

- It is difficult to prove that a set is linearly independent
- Instead work backwards to prove it is linearly dependent
- Ways to prove linear dependency:
  - Set contains vector **0**
  - One set member is a linear combination of others
  - If set is a square matrix it has a determinant 0
- Examples of each proof are shown below:

#### Example:

Consider the following set of vectors:

$$\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$

$$\mathbf{c} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$

$$\mathbf{d} = \mathbf{0}$$

#### Proof of Linear Dependency:

##### Proof 1:

The set of vectors contains **0** so the set is linearly dependent

##### Proof 2: (consider **d** is not in the set)

Find constants so that a linear combination of the vectors have a zero sum:

$$\alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From this we can form simultaneous equations:

$$\alpha + 2\beta - \gamma = 0$$

$$-2\alpha + \beta - \gamma = 0$$

$$\alpha - 3\beta + 2\gamma = 0$$

Solve these to obtain values for the constants:

$$\alpha = -1 \quad \beta = 3 \quad \gamma = 5$$

The possibility of finding values for these constants proves that the set is linearly dependent. If no values of the constants could satisfy the equations then the set of vector **s** would be linearly independent.

##### Proof 3: (consider **d** is not in the set)

Form a square matrix of the set of vectors:

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}$$

Using the expansion method:

$$\begin{aligned} \text{Det}(\mathbf{M}) &= 1 \begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} - 2 \begin{vmatrix} -2 & -1 \\ 1 & 2 \end{vmatrix} - 1 \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} \\ &= 1(2 - 3) - 2(-4 + 1) - 1(6 - 1) = 0 \end{aligned}$$

### 12.5 Spanning Sets

- **Span:** the whole collection of results that can be obtained by making a linear combination
- **Spanning Set:** the set of vectors that span the vector space
- A vector space has an infinite number of spanning sets
- The dimensions of the vector space defines how many independent vectors there are in any spanning set
- E.g. a spanning set for a plane must have 2 independent vectors since a plane is a 2 dimensional vector space

### 12.6 Base Vectors

- **Base Vectors (Basis):** the smallest possible spanning set for a vector space
- There are an infinite number of sets that can form a basis for a vector space
- The dimensions of a vector space defines how many independent vectors there must be in a basis
- E.g. a basis for a plane must be 2 independent vectors that span the plane since a plane is 2 dimensional
- **Dimension:** the number of vectors in a basis of a vector space

### 12.7 Matrix Transformations

- All our transformations are linear transformations
- Take the following linear transformation:

$$\mathbf{A}: \mathbb{R}^5 \rightarrow \mathbb{R}^3$$

- In this transformation a 5 dimensional space is transformed into a 3 dimensional space
- This essentially means:

$$\mathbf{A} \times \mathbf{v} = \mathbf{r}$$

Where **A** is the transformation matrix, **v** is the 5 dimensional space in matrix form and **r** is the resulting 3 dimensional space in matrix form

- We can substitute unknowns to represent the vectors:

$$\begin{pmatrix} a & d & g & j & m \\ b & e & h & k & n \\ c & f & i & l & o \end{pmatrix} \begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \mathbf{r}$$

- This can be opened up and written in another form:

$$\mathbf{r} = v \begin{pmatrix} a \\ b \\ c \end{pmatrix} + w \begin{pmatrix} d \\ e \\ f \end{pmatrix} + x \begin{pmatrix} g \\ h \\ i \end{pmatrix} + y \begin{pmatrix} j \\ k \\ l \end{pmatrix} + z \begin{pmatrix} m \\ n \\ o \end{pmatrix}$$

- This is a linear combination of column vectors proving that the transformation is a linear transformation



• Important terminology:

- **Image vector:**  $\mathbf{r}$  in our scenario
- **Object vector:**  $\mathbf{v}$  in our scenario

### 12.8 The Rank of a Matrix

- A matrix can be considered as a collection of row or column vectors
- Row/column rank of a matrix is the number of independent row/column vectors in the matrix
- Row rank is equal to column rank, both of which are known as the rank of a matrix
- Either can be found by reducing matrix to echelon form:
  - Row Rank: row reduce normally
  - Column Rank: transpose, row reduce, transpose
- Number of non-zero rows (or transposed columns) remaining is the rank of the matrix

### 12.9 Subspaces

- **Subspace:** a subset of vectors in a vector space, that is itself another vector space
- E.g. a 2 dimensional plane is a vector space; a line through it is a 1 dimensional subspace of the plane
- This is because the line is a 1 dimensional vector space
- Ways to (dis)prove the existence of a potential subspace:
  - Linear combination fails to produce a set member
  - Scalar multiplication fails to produce a set member
  - Vector space does not pass through 0
- Similar to proving whether or not a set is a vector space

### 12.10 Special Subspaces

- **Object Space:** the set of linear combinations of object vectors
  - Dimensions: no. of dimensions before transformation
- **Image Space:** the set of linear combinations of image vectors
  - Dimensions: no. of dimensions after transformation
- **Column/Range Space:** the set of linear combinations of the column vectors in a matrix
  - Dimensions: no. of dimensions is equal to the rank
  - Basis: remaining non-zero columns after column reduction
- **Row Space:** the set of linear combinations of the row vectors in a matrix
  - Dimensions: no. of dimensions is equal to the rank
  - Basis: remaining non-zero rows after row reduction

• **Null Space:** the set of vectors that are lost during a transformation

- Dimensions: these are the dimensions that are destroyed during or **nullity** of a transformation

$$n = \text{Rank}(\mathbf{A}) + \text{Nullity}(\mathbf{A})$$

Where  $n$  is the no. of dimensions of the object space

- Basis: refer to the example

#### Example:

A linear transformation,  $\mathbf{A}: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ , is represented by the matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 0 & -3 & 1 & -2 \\ 4 & 5 & 5 & 8 \end{bmatrix}$$

- Find the rank of  $\mathbf{A}$
- Find the dimensions of and a basis for the range space of  $\mathbf{A}$
- Find the dimensions of and a basis for the row space of  $\mathbf{A}$
- Find the dimensions of and a basis for the null space of  $\mathbf{A}$

#### Solution:

##### Part (i):

The rank of the matrix is given by the column rank or row rank. We will use both methods.

##### **Column Rank using Column Reduction:**

Transpose matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 4 \\ 1 & -3 & 5 \\ 3 & 1 & 5 \\ 3 & -2 & 8 \end{bmatrix}$$

Perform normal row reduction:

$$R_1 = R_1 - R_2: \begin{bmatrix} 1 & 3 & -1 \\ 1 & -3 & 5 \\ 3 & 1 & 5 \\ 3 & -2 & 8 \end{bmatrix}$$

$$R_2 = R_2 - R_1: \begin{bmatrix} 1 & 3 & -1 \\ 0 & -6 & 6 \\ 3 & 1 & 5 \\ 3 & -2 & 8 \end{bmatrix}$$

$$R_3 = R_3 - 3R_1: \begin{bmatrix} 1 & 3 & -1 \\ 0 & -6 & 6 \\ 0 & -8 & 8 \\ 3 & -2 & 8 \end{bmatrix}$$

$$R_4 = R_4 - 3R_1: \begin{bmatrix} 1 & 3 & -1 \\ 0 & -6 & 6 \\ 0 & -8 & 8 \\ 0 & -11 & 11 \end{bmatrix}$$

$$R_3 = 3R_3 - 4R_2: \begin{bmatrix} 1 & 3 & -1 \\ 0 & -6 & 6 \\ 0 & 0 & 0 \\ 0 & -11 & 11 \end{bmatrix}$$

$$R_4 = 6R_4 - 11R_3: \begin{bmatrix} 1 & 3 & -1 \\ 0 & -6 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Transpose back to original dimensions:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -6 & 0 & 0 \\ -1 & 6 & 0 & 0 \end{bmatrix}$$

The number of non-zero columns remaining is the rank therefore  $\text{rank}(A) = 2$

**Row Rank using Row Reduction:**

$$A = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 0 & -3 & 1 & -2 \\ 4 & 5 & 5 & 8 \end{bmatrix}$$

$$R_3 = R_3 - 2R_1: \begin{bmatrix} 2 & 1 & 3 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 3 & -1 & 2 \end{bmatrix}$$

$$R_3 = R_3 + R_2: \begin{bmatrix} 2 & 1 & 3 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The number of non-zero rows remaining is the rank therefore  $\text{rank}(A) = 2$

**Part (ii):**

The range space is also known as the column space. The dimensions of the column space is equal to the rank and is therefore 2.

The basis for the range/column space is given by the remaining set of vectors after column reduction:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Note: pull out a factor of 6 from the second vector

**Part (iii):**

The dimensions of the row space is equal to the rank and is therefore 2.

The basis for the row space is given by the remaining set of vectors after row reduction:

$$\{[2 \ 1 \ 3 \ 3], [0 \ -3 \ 1 \ -2]\}$$

**Part (iv):**

The dimensions of the null space are given by the formula:

$$n = \text{Rank}(A) + \text{Nullity}(A)$$

Substitute our values and rearrange:

$$4 = 2 + \text{Nullity}(A)$$

$$\therefore \text{Nullity}(A) = 4 - 2 = 2$$

In order to find a basis for the null space we will need to solve the following equation:

$$Ax = 0$$

Where  $x$  is an unknown matrix

Best method to solve this equation is by using the row reduced form of  $A$ :

$$\begin{bmatrix} 2 & 1 & 3 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this we get 2 equations:

$$2w + x + 3y + 3z = 0$$

$$-3x + y - 2z = 0$$

Because the nullity of  $A$  is 2 we need to find two independent solutions to form a basis:

*First solution:*

If,

$$x = 4 \quad y = 0 \quad z = -6$$

Then,

$$w = \frac{-4 - 3(0) - 3(-6)}{2} = 7$$

Thus solution is:

$$x = 4 \quad y = 0 \quad z = -6 \quad w = 7$$

*Second solution:*

If,

$$x = -1 \quad y = -3 \quad z = 0$$

Then,

$$w = \frac{-(-1) - 3(-3) - 3(0)}{2} = 5$$

Thus solution is:

$$x = -1 \quad y = -3 \quad z = 0 \quad w = 5$$

Thus the basis for the null space is:

$$\left\{ \begin{bmatrix} 7 \\ 4 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ -3 \\ 0 \end{bmatrix} \right\}$$

## 12.11 Solving Linear Equations

- Vector spaces can be used to solve a linear set of equations
- Instead of equating to 0 in the process of finding a basis for the null space of transformation we equate  $Ax$  to a matrix, which will form the equations we need to solve
- Consider the example below:

**Example:**

Solve the equation:

$$Ax = \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}$$

(Use matrices from the example before)

Solution:

When we row reduce, we have to row reduce the whole equation. Basically whatever row operations you use on **A** must be done on the vector on the right hand side as well.

Row reduce the augmented matrix below:

$$\begin{bmatrix} 2 & 1 & 3 & 3 & 5 \\ 0 & -3 & 1 & -2 & 3 \\ 4 & 5 & 5 & 8 & 7 \end{bmatrix}$$

$$R_3 = R_3 - 2R_1: \begin{bmatrix} 2 & 1 & 3 & 3 & 5 \\ 0 & -3 & 1 & -2 & 3 \\ 0 & 3 & -1 & 2 & -3 \end{bmatrix}$$

$$R_3 = R_3 + R_2: \begin{bmatrix} 2 & 1 & 3 & 3 & 5 \\ 0 & -3 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we have two sets of linear equations:

$$2w + x + 3y + 3z = 5$$

$$-3x + y - 2z = 3$$

Find one solution for this using method from before:

$$w = 3 \quad x = -1 \quad y = 0 \quad z = 0$$

The general solution of the equation is then:

$$\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 7 \\ 4 \\ 0 \\ -6 \end{bmatrix} + \mu \begin{bmatrix} 5 \\ -1 \\ -3 \\ 0 \end{bmatrix}$$

The two vectors that are used at the end come from the basis of the null space of the transformation.

# NOTES

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