Proof of Quadratic Formula

Quadratic equations take the form $ax^2+bx+c=0$ (1) with $a\neq 0.$ Any equation of this form may be solved using the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. This formula is proved by completing the square for (1).

Starting from $ax^2+bx+c=0,$ divide through by a to give $x^2+\frac{b}{a}x+\frac{c}{a}=0$ (2) then write

$$x^2+\frac{b}{a}x=\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2=\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}$$

Substitute this into (2) to give $\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}=0$

Now make x the subject. Start by moving the last two terms to the right hand side.

$$\left(x+\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}=\frac{b^2}{4a^2}-\frac{c}{a}$$

Collect the terms on the right by adding them as fractions.

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4c^2}=\frac{b^2-4ac}{4a^2}$$

Square root both sides to give $x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$

Subtraction of $\frac{b}{2a}$ gives $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$