

Geometric Sequences

A geometric (or exponential) progression is any expression formed by multiplying a term by a fixed number to get the next number.

2, 6, 18, 54,...

is a geometric progression. Each term is multiplied by 3 to get the next term.

If a is the first term, and each term is multiplied by r to get the next term, the general geometric progression is

a, ar, ar^2, ar^3, \dots

The n th term in an arithmetic progression is ar^{n-1}

We can also find an expression for the sum of n terms of a geometric series.

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-3} + ar^{n-2} + ar^{n-1} \quad (1)$$

Multiplying by r gives

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-2} + ar^{n-1} + ar^n \quad (2)$$

(1)-(2) gives $S_n - rS_n = a - ar^{n-1}$ because all other terms cancel.

$$\text{Factorising gives } S_n(1 - r) = a(1 - r^n) \rightarrow S_n = \frac{a(1 - r^n)}{1 - r}.$$

We can use these formulae to solve problems.

Suppose the population of rabbits on an island is increasing geometrically. One after they were introduced there are 20 rabbits, and there are also 4 sets of rabbit bones. Find how many rabbits were introduced and the ratio of the total number of rabbits alive or dead from year to year.

20 rabbits in year 1: $ar = 20$

$$24 \text{ rabbits alive or dead in year 1: } \frac{a(1 - r^2)}{1 - r} = \frac{a(1 - r)(1 + r)}{1 - r} = a(1 + r) = a + ar = 24$$

Subtract the first of these from the second to give $a = 4$

Substitute this into the first equation $ar = 20$ to give $r = 4$

The ratio between populations in successive years is $1 : 4$ - note there are 20 live rabbits and 4 dead ones in year 2.