Completing the Square

Completing the square makes it possible to find the maximum of minimum value of a quadratic function without sketching it., or to solve quadratic equations without using the quadratic formula.. We start with an expression $p(x)$ to express in the form $a(x+b)^2+c$

We might multiply this out to obtain $ax^2+2\text{abx}+ab^2+c$. Now we equate $p(x)$ to this and solve for the coefficients a, b, c.

Example: Complete the square for the expression $3x^2+12x-5$:

$3x^2+12x-5=ax^2+2\text{abx}+ab^2+c$

Equating coefficients of $x^2: 3x^2=ax^2 \rightarrow a=3$

Equating coefficients of $x: 12x=2\text{abx} \rightarrow 12=2\text{ab} \rightarrow b=\frac{12}{2a}=\frac{12}{2*3}=2$.

Equating constant terms: $-5=ab^2+c=3*2^2+c=12+c \rightarrow c=-5-12=-17$

Hence $3x^2+12x-5$ in completed square form is $3(x+2)^2-17$

Having completed the square we can solve the equation $3x^2+12x-5=0$:

$$3(x+2)^2-17=0 \rightarrow 3(x+2)^2=17 \rightarrow (x+2)^2=\frac{17}{3} \rightarrow (x+2)=\pm\sqrt{\frac{17}{3}} \rightarrow x=-2\pm\sqrt{\frac{17}{3}}.$$

Example: Complete the square for the expression $2x^2-10x+1$. Hence solve the equation $2x^2-10x+1=0$

$2x^2-10x+1=ax^2+2\text{abx}+ab^2+c$

Equating coefficients of $x^2: 2x^2=ax^2 \rightarrow a=2$

Equating coefficients of $x: -10x=2\text{abx} \rightarrow -10=2\text{ab} \rightarrow b=\frac{-10}{2a}=\frac{-10}{2*2}=-2.5$.

Equating constant terms: $1=ab^2+c=2*-2.5^2+c=12.5+c \rightarrow c=1-12.5=-11.5$

Hence $2x^2-10x+1$ in completed square form is $2(x-2.5)^2-11.5$.

Now we can solve $2x^2-10x+1=0$:

$$2(x-2.5)^2-11.5=0\rightarrow 2(x-2.5)^2=\frac{23}{2}\rightarrow (x-2.5)^2=\frac{23}{4}\rightarrow (x-2.5)=\pm\sqrt{\frac{23}{4}}\rightarrow x=2.5\pm\sqrt{\frac{23}{4}}.$$