A game theoretic analysis of a variety of two-person zero-sum poker models

"Industry executives and analysts often mistakenly talk about strategy as if it were some kind of chess match. But in chess, you have just two opponents, each with identical resources, and with luck playing a minimal role. The real world is much more like a poker game, with multiple players trying to make the best of whatever hand fortune has dealt them. In our industry, Bill Gates owns the table until someone proves otherwise." -David Moschella, *"Computerworld"*

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Abstract

Poker has for a long period been a topic demonstrating the application of game theory in literature. In this essay I will make game theoretic analyses of several different versions of simple poker models. These models can, importantly, be applied to illustrate possible scenarios in more multifaceted poker. An example of such a scenario would be in the *end game*[1] – where all *hands* have become determined. Game theory is particularly applicable to this stage in the game, due to the information available regarding the probability that a particular hand will win (i.e. one hand is of a higher value than the opposing one), allowing the derivation of specific strategies. I will limit the scope of the essay to two-person (zero-sum) situations, allowing me to effectively look at a greater variety of models – as the analyses for two-person games are much less complex than for multiple-player ones where the possibility of collusion is existent . Poker lends itself suitably to the application of game-theory due to the elements of *bluffing*, probability, and strategy. In this paper I will derive *optimal* strategies for both players, illustrating that different strategies should be adapted to different models and players (i.e. player 1 or player 2), whilst indicating that in order to win – in the long run – you must, ironically, deviate from playing optimally.

[Word Count: 218]

[1] Key words are initially put in italics and are defined in the appendix.

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Acknowledgements

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1. Introduction

1.1. Why did I choose poker for a game theoretic analysis?

Poker is a card game that I enjoy playing as it involves many areas of mathematics. It entails a complex mix between probability and strategy, offering an interesting game theoretical analysis. Poker is essentially a *multi-player non-deterministic zero-sum game with imperfect information* fitting into the range of problems that game theory can be applied to solve. It is a sequential move game where the actions of each player are a dependent response to observed variables such as *bets* and reactions. Each individual player has no definite information about other hands, only their predictions based upon what they manage to extract from a player's leaks and responses. Poker is a competitive game where the total sum of utility gained by the players is zero despite which strategy is employed by them. Thus it is classified as a zero-sum game; in other words the losses of one player are the winnings of another player.

1.2. What is Game Theory?

Game theory "is the study of the ways in which *strategic* interactions among *rational players* produce *outcomes* with respect to the preferences of those players, none of which might have been intended by any of them"[1]. It can be applied to interdependent actions of individuals, groups, firms, or any combination of these, allowing them to come to agreements or make choices resulting in optimal outcomes. However in this essay I will look at its original use; games. I will use game theory, as it was initially, to find optimal strategies in different two-player zero-sum poker games.

[1] See reference < http://plato.stanford.edu/entries/game-theory/>

2. The strategic form[2]

Any two-person zero-sum game can be represented in strategic or *extensive form*. The *strategic form* of a game represents different strategies and their corresponding *payoffs* in a matrix. It allows an easy analysis; however it is a compact method of showing a game, where certain information sets are not represented. It is therefore usually used to represent simultaneous move games, where the players have no information about the other player. When a game is represented in strategic form we must remove any *dominated strategies* in order find *mixed strategies* for each player. However, this is merely possible where there are only a few permissible pure strategies. We will look at such cases when solving poker models where the possible hands of each opponent are fixed.

An example of a game in a reduced strategic form is:

$$\begin{array}{cc} & \text{Player II} \\ & \begin{array}{cc} A & B \end{array} \\ \text{Player I} \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \end{array}$$

3. The extensive form

3.1. Game Tree[3]:

Poker is usually represented by a tree or in extensive form and then reduced to a strategic form when deriving mixed strategies. In the extensive form concepts such as bluffing and signalling are revealed, demonstrating a sequential move game. The extensive form is a game tree, which is essentially a *directed graph*. There is an initial vertex indicating the beginning of the game and terminal vertices indicating its end. Between the initial vertex

[2] Solving games in strategic form will be shown in part 5.
[3] An explanatory diagram of such a tree is provided on page 4.

and a terminal vertex there exists one unique path; thus there are no loops and all vertices are connected to each other through a path or branch.

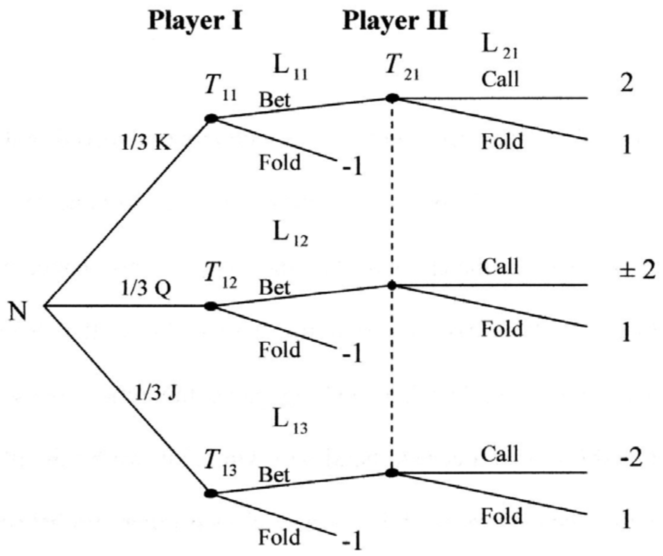
3.2. Chance moves:

In poker the first move or the initial vertex represents the chance move involved with the dealing and shuffling of the cards. Each player has no control of what cards he receives and no, or limited, information regarding the cards of the other player. This chance move is often referred to as nature (N). It is assumed that the players know the different probabilities of chance moves; for example when making game theoretic analyses of poker we assume that the probability that a certain hand will win is known by the player holding that hand; however the players are not fully aware of each others probabilities. Therefore poker is considered a game of imperfect information.

3.3. Information sets:

In poker, where the first move is a chance move, the players are aware of a limited outcome, but not the entire outcome. When player I acts he knows what cards he has or on what vertex he is, however player II does not have this information, since he is not aware of the cards of player I. Thus, player I has as many information sets as possible hands and player II will have one information set indicated by a dotted line or circle drawn around the possible vertices. This suggests the possibility of bluffing, or purposely signalling to the other player a weaker or stronger hand than you really have.

An illustrative diagram of a game in extensive form with relevant labels[4]:



Any finite two-person zero-sum game in extensive form will have[5]:

- A finite tree with a set of vertices
- A pay-off, represented by a real number, at each terminal vertex[6]
- A nature (N) consisting of non-terminal vertices (indicating points where chance moves occur) including an appropriate probability distribution along the diverging branches
- Two groups of information sets denoted by $T_{11}, \ldots, T_{1k_1}$ (for player 1) and $T_{21}, \ldots, T_{2k_2}$ (for Player II)
- For each information set there exists a corresponding set of labels (actions) $L_{11}, \ldots, L_{1k_1}$ (for player I) and $L_{21}, \ldots, L_{2k_2}$ (for player II)

[4] This diagram is taken from my primary model, where K Q J represent King, Queen and Jack respectively.
[5] Definitions are adapted from: Ferguson, T. *Game Theory, Part II, Two-person zero-sum games.p.48.*
[6] Note that it is the pay-off for player I that is shown on the terminal vertices, thus the negative pay-off for player II, since it is interpreted as a zero-sum game.

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4. Moving from extensive to strategic form

4.1. Finding the number of pure strategies[7]:

When a game is represented in extensive form we must first consider the different sets of pure strategies (i.e. bet, *call*, or fold) telling the players what actions to make in their information sets. We let X denote the set of all the pure strategies of player I and Y the equivalent for player II. If $T_{11}, \ldots, T_{1k_1}$ represents the different information sets for player I and $L_{11}, \ldots, L_{1k_1}$ represents the corresponding sets of labels (actions) then there will be a k_1-tuple number of pure strategies such that $X = \{x, \ldots, x_{k_1}\}$ where $x \in L_{1i}$ for each *i.* If each L_{1i} consists of m_i elements there will be a k_1-tuple or $m_1 m_2 \ldots m_{k_1}$ number of strategies. This can be summarized by saying that the number of pure strategies making up the set X is given by $m_i^{\,k_1}$ (where k_1 indicates the number of information sets). Similarly, to find the set, Y, of pure strategies of player II, we let $T_{21}, \ldots, T_{2k_2}$ represent the information sets of player II and $L_{21}, \ldots, L_{2k_2}$ represent the corresponding sets of labels. Thus, there will be a k_2-tuple number of pure strategies such that $Y = \{y, \ldots, y_{k_2}\}$ where $y \in L_{2i}$ for each j. If n_j denotes the number of elements in each L_{2i} then player II will have $n_1 n_2 \ldots n_{k_2}$ or $n_j^{\,k_2}$ number of pure strategies in the set Y.

4.2. Expected Values:

When calculating the payoffs of different strategies, in game theory, we will replace the individual payoffs, at the terminal vertices, by an average expected value, due to the use of mixed strategies resulting from chance moves. If x and y, where $x \in X \quad y \in Y$,

[7] The method and general subscripts are adapted from: Ferguson, T. *Game Theory, Part II, Two-person zero-sum games.p.50.*

represent certain pure strategies for players I and II respectively, the payoff of the strategies will be a random quantity (depending on the value of the hands). We therefore replace the different possible payoffs by an average value we denote $A(x,y)$.[8]

When moving from an extensive form to a strategic form we replace the information sets by changing payoffs at the terminal vertices with $A(x,y)$ for the corresponding strategies in a matrix.

5. Optimal strategies for simplified one-card poker models

5.1. Unitary bet-size:

Consider the simplified version of poker, "one card poker", where there is a deck consisting of three cards; a Jack, a Queen, and a King. Each player – there are two – places 1 monetary unit into the pot before starting; this is known as an *ante*. The players are then dealt one card, face-down. Player I, the player to the left of the *dealer*, may then fold or bet. If he decides to fold he will lose 1 unit (his ante) and player II will subsequently win this unit. If player I decides to bet he may bet 1 unit, which player II will call unless he chooses to fold. There will be a *showdown* if player II calls, where the winner – the one with the highest card – wins the pot. Even though the pot will at that stage consist of 4 units the winning player is only gaining half of it; as half of the pot contains his money. Thus we will not consider the money placed in the pot as a sunk cost. If player II chooses to fold, losing 1 unit, player I wins 1 unit respectively, as it is a zero-sum game where one player's loss is another's gain. We can represent this game in extensive form:

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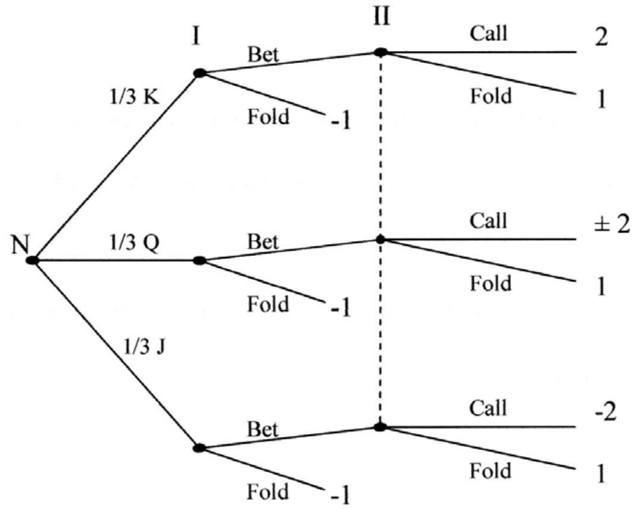
[8] An example where average values are calculated occurs on page 8.



Using the information provided by the tree we can formulate several different strategies and represent them in a strategic form allowing us to get rid of dominated strategies.

Consider all of the available strategies for player 1. Player I has three different information sets and two different elements in each label and therefore has $2^3 = 8$ pure strategies. These strategies include:

1 Bet with any card.
2 Bet with King or Queen and fold with Jack.
3 Bet with King and fold with Queen or Jack.
4 Bet with King or Jack and fold with Queen.
5 Bet with Jack and fold with King or Queen.
6 Bet with Queen or Jack and fold with King.
7 Bet with Queen and fold with King and Jack.
8 Always fold.

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Where X represents all pure strategies; $X = \{1,2,3,4,5,6,7,8\}$

Some of these strategies are obviously better than others, but in this example I will consider them all to demonstrate how we can clearly decide which strategies to eliminate.

Player II does not have as many strategies as Player 1, as he only has one information set, thus he will have $2^1 = 2$ pure strategies:

1' If player I bets call

2' If player I bets fold

Where Y represents all pure strategies; $Y = \{1',2'\}$

To represent the strategies in strategic form we will begin by calculating the expected value for each of the player's strategies, in terms of player I:

If player II always calls the results of player I's corresponding strategies or $A(x, 1')$ are as follows:

1 1/3 x 2 + 1/6 x 2 + 1/6 x -2 + 1/3 x -2 = 0

2 1/3 x 2 + 1/6 x 2 + 1/6 x -2 + 1/3 x -1 = 1/3

3 1/3 x 2 + 1/3 x -1 + 1/3 x -1 = 0

4 1/3 x 2 + 1/3 x -2 + 1/3 x -1 = -1/3

5 1/3 x -2 + 1/3 x -1 + 1/3 x -1 = -4/3

6 1/6 x 2 + 1/6 x -2 + 1/3 x -2 + 1/3 x -1 = -1

7 1/6 x 2 + 1/6 x -2 + 1/3 x -1 + 1/3 x -1 = -2/3

8 1/3 x -1 + 1/3 x -1 + 1/3 x -1 = -1

If player II always folds the results of the corresponding strategies of player I will be:

1 1/3 x 1 + 1/3 x 1 + 1/3 x 1 = 1

2 1/3 x 1 + 1/3 x 1 + 1/3 x -1 = 1/3

3 1/3 x 1 + 1/3 x -1 + 1/3 x -1 = -1/3

4 1/3 x 1 + 1/3 x 1 + 1/3 x -1 = 1/3

5 1/3 x 1 + 1/3 x -1 + 1/3 x -1 = -1/3

6 1/3 x 1 + 1/3 x 1 + 1/3 x -1 = 1/3

7 1/3 x 1 + 1/3 x -1 + 1/3 x -1 = -1/3

8 1/3 x -1 + 1/3 x -1 + 1/3 x -1 = -1

In each case, Call and Fold, the column on the left represents player I's winnings and the column on the right represents player II's winnings respectively.

$$
\begin{array}{cc}
 & \text{Player II} \\
 & \begin{array}{cc} \text{Call} & \text{Fold} \end{array} \\
\text{Player I} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\begin{pmatrix}
0 & 0 & 1 & -1 \\
1/3 & -1/3 & 1/3 & -1/3 \\
0 & 0 & -1/3 & 1/3 \\
-1/3 & 1/3 & 1/3 & -1/3 \\
-4/3 & 4/3 & -1/3 & 1/3 \\
-1 & 1 & 1/3 & -1/3 \\
-2/3 & 2/3 & -1/3 & 1/3 \\
-1 & 1 & -1 & 1
\end{pmatrix}
\end{array}
$$

where the numbers along the outer-left-side of the matrix represents player I's different strategies

Eliminating the dominated strategies, 3-7, leaves us with strategies 1 and 2 for player I:

$$
\begin{array}{ccc}
 & & \text{Player II} \\
 & & \begin{array}{cc} \text{Call} & \quad\quad \text{Fold} \end{array} \\
\text{Player I} & \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 0 & 1 & -1 \\ 1/3 & -1/3 & 1/3 & -1/3 \end{pmatrix}
\end{array}
$$

To find the unique optimal strategy for each player:

Let p represent the frequency or probability in which player I employs strategy 1 and (1-p) the frequency in which he chooses strategy 2. For player II q will denote the frequency in which he chooses to call and (1-q) the frequency at which he should fold: Where p and q are both restricted to the domain: $0 \leq 1$.

These values must make each player's optimal strategy indifferent to the opponent; thus:

$0p - 1/3(1 - p) = -p - 1/3(1 - p)$ ……(1)

$0q + 1(1 - q) = 1/3\, q + 1/3(1 - q)$ ……(2)

From (1) we get that p must be equal to zero. Therefore player I should always consider strategy 2, where he always bets with a King or a Queen and always folds with a Jack. From (2) we can conclude that q is equal to 2/3. Player II should therefore call 2/3 of the time and fold 1/3 of the time. If we want we can find the two cards that player II should always call with and the card he should always fold with by simply considering his different calling and folding strategies like we did with the betting and folding strategies of player I. However in this case it is obvious that player II should never call with a Jack,

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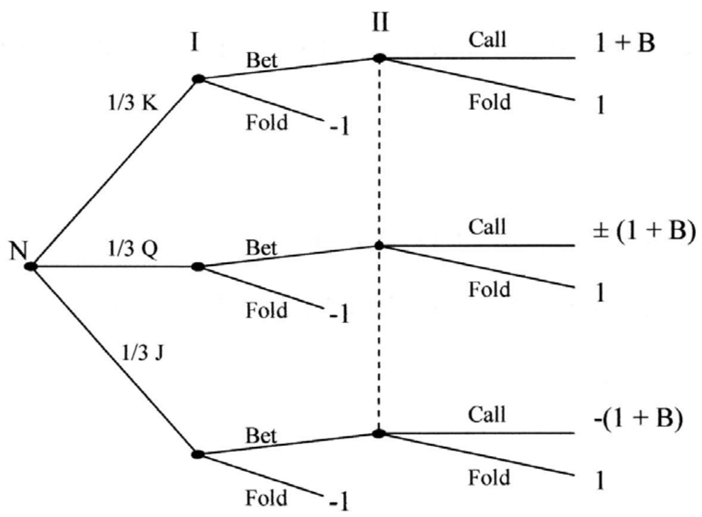
as he is guaranteed to lose 2 units if he does as opposed to 1 unit if he folds. Player II should, therefore, always call with a King and a Queen and fold with a Jack.

This allows us to calculate the expected value of the game for each player. For player I it will be 0 x 0 + 1/3 x 1 = 1/3 and for player II: 2/3 x 0 + -1 x 1/3 = -1/3 (the opposite of player I's gain as it is a zero-sum game). This indicates that the game favours player I, he will on average win 1/3 more unit per round.

5.2. Choosing the optimal bet size:

We can take a look a simple model (similar to the one in part 5.1.) and investigate how we can alter the bet size to player I's advantage depending on which card he holds. The model represented in extensive form will be:



Using information from the previous question, we will ignore any perverse strategies and, solely consider the admissible ones denoted by x, where $x \in X$ and X represents a set of the total number of pure strategies.

1 Bet with any card

2 Bet with King or Queen and fold with Jack

Thus $x = \{1,2\}$

Since player II only has one information set and two actions (i.e. call or fold) the set, Y, will consist of $2^1 = 2$ pure strategies; hence $Y = \{1',2'\}$:

1' Call

2' Fold

The expected value for player I's different strategies, if player II chooses to always call, $A(x, 1')$, will be:

1 1/3 (1+B) + 1/6 (1+B) – 1/6 (1+B) – 1/3 (1+B) = 0

2 1/3 (1+B) + 1/6 (1+B) – 1/6 (1+B) – 1/3 x 1 = 1/3(1+B) – 1/3 = 1/3 B

…and $A(x, 2')$ will be:

1 1/3 x 1 + 1/3 x 1 + 1/3 x 1 = 1

2 1/3 x 1 + 1/3 x 1 + 1/3 x -1 = 1/3

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Since we are treating the game as a zero-sum game player II's expected pay-off will be the negative of player I's:

$$\begin{array}{ccc} & & \text{Player II} \\ & & \begin{array}{cc}\text{Call} & \text{Fold}\end{array} \\ \text{Player I} & \begin{array}{c}1\\2\end{array} & \begin{pmatrix} 0 & 0 & 1 & -1 \\ B/3 & -B/3 & 1/3 & -1/3 \end{pmatrix} \end{array}$$

We denote p as the probability of player I employing strategy 1 and $1 - p$ the probability that player I employs strategy 2, where $0 \le p \le 1$.

In order to make player II indifferent: $0p - B/3\,(1 - p) = -p - 1/3\,(1 - p)$ thus;

$-B/3\,(1 - p) = -2/3\,p - 1/3 \;\therefore\; -B(1 - p) = -2p - 1 \;\therefore\; B = \dfrac{2p+1}{1-p}$ where $p \neq 1$ … (1)

From this we can also derive that $p = \dfrac{B-1}{B+2}$ $=$

$$\begin{array}{r} 1 \\ B+2\overline{\big)\,B-1} \\ -B-2 \\ R=-3 \end{array}$$

$\therefore\; p = 1 - \dfrac{3}{B+2}$

As $B \to \infty$ $p \to 1$, thus as the size of the bet increases player I will choose to bet more often. However, it is important to note that player I will never limit his strategy to 1, this means that he always bets, as player II would realize and take advantage of this information. We can also use this equation, (1), to determine values of p for a known bet size, i.e. at $B = 1$ $p = 0$ (see model in part 5.1.); in other words player I would never bluff with a bet size of 1 unit. Thus with larger bets player I will bluff more often as this provides a greater negative incentive for player II to call; the risk is too big.

If we find the probability that player II will call a bet, q, in terms of B, we can calculate the optimal bet size, resulting in the biggest expected value, when player I has a King, Queen, or Jack.

We will let q represent the probability that player II will call and $1 - q$ the probability that he will fold, where $0 \leq q \leq 1$.

In order to make player I indifferent between 1 and 2: $0q + (1 - q) = B/3\ q + 1/3\ (1 - q)$

thus; $3(1 - q) = Bq + (1 - q) \therefore\ 3 - 3q = Bq + 1 - q \therefore Bq = 2 - 2q \therefore\ B = 2/q - 2$

$\therefore\ q = \dfrac{2}{B+2}$ where $B > 0$ ……….(2)

If player I has a king then his expected value, A, will be (if he chooses to bet):

$(1+B)q + 1\ (1 - q)$ if we substitute q with $\dfrac{2}{B+2}$ (2) we get $\dfrac{2(1+B)+B}{B+2} = \dfrac{3B+2}{B+2}$

As $B \to \infty$ $A \to \dfrac{3B}{B} \to 3$ he should therefore bet as much as possible.

When finding the optimal bet for the Queen we should consider what the other player might have; a King or Jack. Player II will obviously always call with the King, but never with the Jack, thus $q = 1/2$ giving a value of 2 for B. This value, is the optimal for player II, as it makes player I indifferent between folding and betting. Therefore player I should try to deviate from it as much as possible:

Player I can either bet or fold. If he bets then his expected value should be greater than -1, thus $-(1+B)1/2 + 1/2 > -1 \therefore B < 2$ As $B \to 0$ $A \to 0$, player I should therefore bet the smallest amount possible if less than 2 when he has the Queen.

The Jack determines the most interesting betting method. The expected value for player I, of betting, can be represented by: $-(1+B)q + (1 - q)$ if we substitute q with $\dfrac{2}{B+2}$ (2) we

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get $\frac{-2(1+B)+B}{B+2}$ which can be simplified to $\frac{-(2+B)}{B+2}$ or -1. Thus the pay-off with a Jack will be unaffected by changes in the bet size. However, knowing that player II will either have a Queen or a King, we can derive an optimal strategy where player I only bets 1/2 of the time (as he is indifferent to folding and betting when player II has the King). When he bets he should therefore choose to bet as high as possible (as with the King) reducing the probability, q, that player II calls; resulting in a pay-off of 1. This also prevents player II from distinguishing between the different information sets (i.e. he can not be sure if player I has a King or Jack judging from the betting).

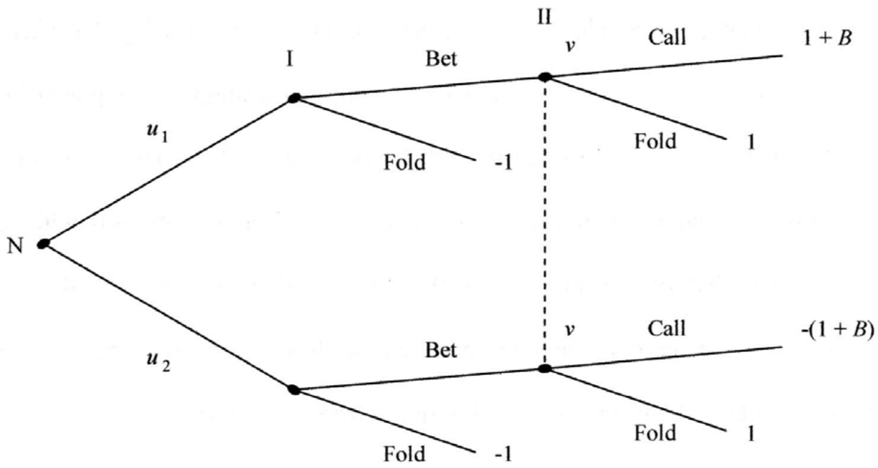
6. Solving games where the values of different hands are represented by an interval

6.1. With a variable bet size:

I am now going to look at a model where each player – after anteing one compulsory unit each into the pot –receives a hand[9] expressed on the interval $[0,1]$. The worst combination of cards, or hand, is represented by 0 and the best possible hand is represented by 1. The value of player I's hand is denoted by u, a random variable of U, and the value player II's hand, is similarly, denoted by v, a random variable of V. Both U and V are uniformly distributed on $[0,1]$. If $u > v$ then player I wins if there is a showdown. Player I will be the first to act, having the choice of betting any real positive bet given by B or folding. If he chooses to bet, then player II will either call – resulting in a showdown where the hand of greatest value wins the accumulated pot – or fold.

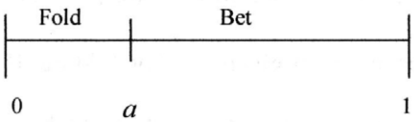
[9] The number of cards in the hand is not of interest in this part, and therefore ignored.

The game tree for this model – the Borel model – can be drawn as:

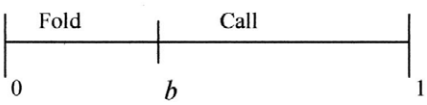


In order to solve this game we will state that player I will decide to bet if $u > a$, where a is an undefined constant on the interval $[0,1]$, and fold otherwise. Similarly player II will call a bet if $v > b$, where b is an undefined constant on the interval $[0,1]$, and fold otherwise. Using a continuous probability density function we can find the values for a and b, allowing us to derive a general optimal strategy for each player.

Player I:



Player II:



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We can find the values for a and b, using a continuous probability density function, allowing us to derive a general optimal strategy for each player.

If player I chooses to bet then the expected value for player II – if he chooses to call – will be $(1 + B)w - (1 + B)(1 - w)$, where w represents the probability that player II will win. In order for it to be worth calling $(1 + B)w - (1 + B)(1 - w) \geq -1$, as player II will ultimately lose 1 if he folds, disregarding the value of his hand.

Simplifying $(1 + B)w - (1 + B)(1 - w) \geq -1$ gives us $w \geq \dfrac{B}{2(B+1)}$

Thus if: $w \geq \dfrac{B}{2(B+1)}$ player II should call.

We must now find an expression for w indicating the probability that $v > u$. If player I chooses to bet, u will, subsequently, lie on the interval $[a,1]$. Player II will win when u lies on $[a,v]$, the probability of that is w.

Therefore $w = \displaystyle\int_a^v \frac{du}{1-a} = \frac{v}{1-a} - \frac{a}{1-a} = \frac{v-a}{1-a}$ where $a \leq v \leq 1$

If we substitute w with $\dfrac{v-a}{1-a}$ where $a \neq 1$ in the equation $w \geq \dfrac{B}{2(B+1)}$ then

$\dfrac{v-a}{1-a} \geq \dfrac{B}{2(B+1)}$ $\therefore$ $2(v-a)(B + 1) \geq B(1 - a)$

(since a and v are both restricted to the range defined by $[0,1]$ and $B > 0$)

$\therefore$ $v(B + 1) \geq \dfrac{B - Ba + 2Ba + 2a}{2}$

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Thus in order for player II to call $v \geq \frac{B+Ba+2a}{2(B+1)}$. Player II will win a hand if he calls with $v > \frac{B+Ba+2a}{2(B+1)}$ or similarly $v > b$. Therefore we can say that the value b, lying on the interval $[0,1]$, is $\frac{B+Ba+2a}{2(B+1)}$.

Player I will lose if he decides to bet with a hand who's value is represented by $u \leq b$ since player II will only call when $v > b$. If player I chooses to bet then the probability that he will win, 1 unit, – as player II will fold when $v \leq b$ – is b; and he will lose $(1+B)$ units with a probability of $(1 - b)$. Hence his average pay-off from betting, is thus, presented by: $-(1+B)(1 - b) + b$. If player I chooses to fold his average pay-off will be 1. Player II will want to make player I indifferent between betting and folding; therefore he will adjust his play such that $-(1+B)(1 - b) + b = -1 \therefore Bb + 2b - B - 1 = -1 \therefore b = \frac{B}{B+2}$ (as B increases b increases; thus at larger bet sizes player II will call less frequently.)

Hence, if we substitute our value for b with $\frac{B+Ba+2a}{2(B+1)}$ we derive

$$\frac{B+Ba+2a}{2(B+1)} = \frac{B}{B+2} \therefore B + a(B+2) = \frac{2B(B+1)}{B+2} \therefore a = \frac{2B(B+1) - B(B+2)}{(B+2)^2}$$

Thus $a = \frac{B^2}{(B+2)^2}$ (as B increases a increases; thus at larger bet sizes player I will bet less frequently.)

From these calculations we have derived that $a = \dfrac{B^2}{(B+2)^2}$ and $b = \dfrac{B}{B+2}$. This means that:

Player I will choose to bet with a probability of $1 - \dfrac{B^2}{(B+2)^2} = \dfrac{4(B+1)}{(B+2)^2}$

Player II will call with a probability of $1 - \dfrac{B}{B+2} = \dfrac{2}{B+2}$

When there is a showdown (player I bets and player II calls) player II's probability of having a better hand can be determined by considering the difference between the range of $[a,1]$ and $[b,1]$. Thus, $\dfrac{4(B+1)}{(B+2)^2} - \dfrac{2}{B+2} = \dfrac{2B}{(B+2)^2}$ represents the probability that player I will lose when he decides to bet, which he does at a probability of $\dfrac{4(B+1)}{(B+2)^2}$.

Player I will, when he chooses to call, lose (1+B) units with a probability of $\dfrac{2B}{(B+2)^2} \div \dfrac{4(B+1)}{(B+2)^2} = \dfrac{2B}{4(B+1)}$ and $1 - \dfrac{2B}{4(B+1)} = \dfrac{2(B+2)}{4(B+1)}$ of the time have an equal probability as that of player II in winning or losing (1+B) units, yielding an expected value of 0.

Thus, from this information, we can calculate player I's overall pay-off when both players are playing optimally:

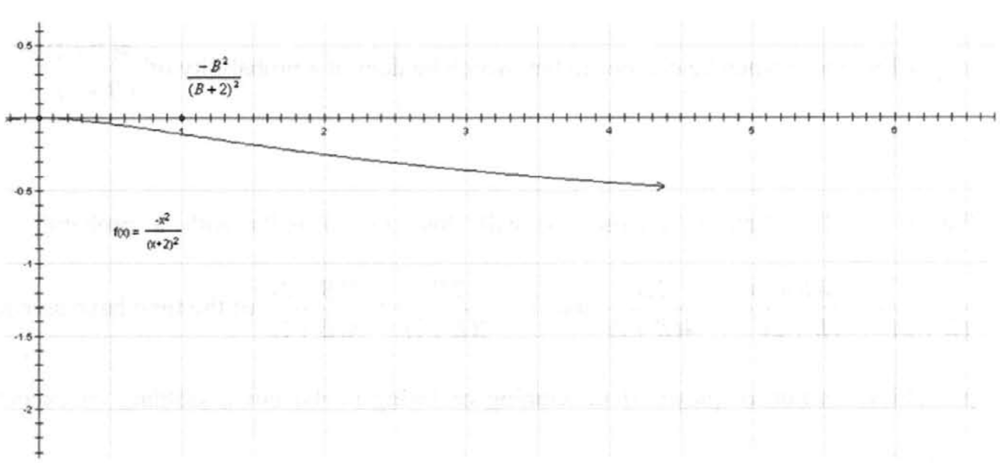
$$\frac{B^2}{(B+2)^2} \times -1 + \frac{4(B+1)}{(B+2)^2}\left[\frac{B}{B+2} \times 1 + \frac{2}{B+2}\left(\frac{2B}{4(B+1)} \times -(1+B) + \frac{2(B+2)}{4(B+1)} \times 0\right)\right]$$

$$= \frac{-B^2}{(B+2)^2} + \frac{4(B+1)}{(B+2)^2}\left[\frac{B}{B+2} + \frac{2}{B+2}\left(\frac{-2B(1+B)}{4(B+1)}\right)\right]$$

$$= \frac{-B^2}{(B+2)^2} + \frac{4(B+1)}{(B+2)^2}\left[\frac{B}{B+2} + \frac{-B}{B+2}\right] = \frac{-B^2}{(B+2)^2}$$

Thus the expected value for player I is: $\frac{-B^2}{(B+2)^2}$

He will lose an amount equal to the probability that he chooses to fold (a). Player I's expected value will always be negative when B >0, we can plot the function demonstrating the expected value of player I:



Player II's expected value will be given by $\frac{B^2}{(B+2)^2}$ or a (as it's a zero-sum game), which will be the reflection of the line, showing the expected value for player I, in the x-axis. Player I will minimize his expected loss (and player II's gain) by choosing to bet less, thus he should bet as little as allowed. However this is solely a strategy to minimize losses, when playing optimally.

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As we previously derived, player II will only call a bet if his winning probability (w) is represented such that:

$$w \geq \frac{B}{2(B+1)}$$

Thus the larger bet player I chooses the less likely it is for player II to call. This may be an admissible strategy if player I has a hand normally not worth betting with and predicts that the value of player II's hand will induce him to fold, allowing player I to win the antes. However this, in the theoretical model I have looked at, is not optimal since player II has not had the opportunity to reveal any information about his hand, making player I's strategy unjustified. In practice, however, this strategy may be effective due to the increased information through body language and past moves. This is an important difference between theory and practice and the respective optimal and *maximal* players, which I will discuss in my conclusion.

7. Conclusion

"*Poker is the only game where it is correct to play incorrect*" –David Sklansky

In the poker models, that I have analysed, game theory has been applied to produce optimal strategies. An optimal player will effectively result in the best pay-off against a perfect opponent. It is important, however, to note that you can not win in the long run with optimal strategies, as then cards and position (i.e. player I or II) will alternate resulting in an overall expected value of zero (as it's a zero-sum game). Optimal strategies merely show how to play against an optimal player, but do not exploit apparent mistakes. If you use an optimal strategy you will be indifferent to any of your opponent's actions, allowing you to have a constant expected value. When your opponent does not play optimally then you should – in order to maximize your pay-off – play maximally.

You should take advantage of un-optimal plays from your opponent. To maximize your pay-off you can induce your opponent into playing sub-optimally by, for example, leaking misleading information. In order to play maximally you must deviate from playing optimally allowing you to exploit any observed weaknesses in your opponents play. In doing so you subsequently take the risk of playing sub-optimally, however this risk can be justified through awareness and the potential increased expected value from taking it. Since all humans make mistakes the best strategy involves playing optimally until you have noticed any weaknesses or patterns in your opponents play and then deviating from optimality to take advantage of these observations. However this approach is only possible during long term interactions. Game theory is still applicable to real-world poker when determining bet sizes and making decisions based on imperfect sequential information (i.e. when to call).

7.1. Application:

The models of poker that I have considered do not represent full-scale poker games; however I have demonstrated that the application of game theory can be used to produce optimal strategies and bet sizes when the probability that a certain hand will win is known. This can be applied to the last betting round – end game – of any poker variant, as the probability of a winning hand can be calculated, since no further cards will be revealed and no changes in hands will occur. This allows you to compare your hand strength with the existing, possible, interval. However I have not considered the action of *checking* or the presence of more than two players. I have solely shown the use of game theory in two-person zero-sum games to derive different mixed strategies.

Appendix

History and uses of game theory

Game theory has become increasingly useful and popular as its broad applications and potential are being explored. The earliest trace of game-theory can be dated back to 1838, where Antoine Cournot used game-theoretic approaches to his study of duopolies. Game theory was later re-introduced by Emile Borel in 1921. His work on the theory of games encouraged the mathematician, Jon von Neumann, in the paper on the "theory of parlour games" in 1928. It was von Neumann who first, officially, introduced game theory as an individual field of mathematics. Von Neumann showed that for two-person zero-sum games an optimum strategy could always be found. He later extended his analysis to games of multiple players, realizing the impact of coalition. Together with the Austrian mathematician Oskar Morgenstern he wrote a book called *The Theory of Games and Economic Behaviour*, where different applications of game theory were discussed. During the 1950's and 1960's the concept of non-cooperative game theory was broadened, allowing the application to both politics and war. John Nash was one of the key mathematicians encouraging this extended application of game theory; in 1950 he showed that most finite games will have an equilibrium point – Nash equilibrium – resulting in the optimal outcome regarding any of the opponent's choices. Now game theory has a greater variety of uses including anything from negotiations to oligopolies. It is applicable to most academic fields and is becoming increasingly useful in artificial intelligence; for example, video poker *bots* will be programmed using game theory.

Playing Poker

Poker is played with a standard deck of 52 cards. There are two main variants stud and hold'em, however I will not go into any description regarding how each specific variant is played, I will solely look at the poker relevant to this essay. In the two-person models I will be looking at (also known as *heads-up*) player I and II each ante 1 unit into the *pot* before starting the game. This is to encourage betting later on. Player II shuffles a deck (in some models only a few cards) and proceeds to deal a certain number of cards to player I and himself in a clockwise fashion. After the players have looked at their cards, player I will choose to either bet (where he places a certain unit in the pot) or fold (where he chooses not to bet and therefore forfeits). If he folds then player II will win the antes, whereof one is his. Since we are not considering what is placed in the pot as a sunk cost, player II therefore gains one unit whilst player I loses one unit. However if player I, instead, chooses to bet player II must either fold, call, or *raise*. If he folds then player I will win the antes, and if he calls then there is a showdown, where the hands of each player is shown and the player with the highest hand wins the pot. However player II can also choose to raise the bet of player I (simply by betting more); where player II can either fold, where the pot is won to player I, or call, the difference between the raise and his previous bet, resulting in a showdown. It is important to note that in order for there to be a showdown each player must have contributed an equal amount to the pot. After a hand is played the dealer will alternate, thus player I will now be the dealer causing the betting order to shift respectively.

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Definitions

Antes:

Antes are a fixed amount each player must contribute to the pot before starting a game.

Bet:

This is a chosen value a player places in the pot at the start of a betting round.

Bluff:

A bluff is when you deliberately leak misleading information (i.e. betting strongly with a weak hand) with purpose of benefiting.

Bot:

Poker bots are programs that model optimal players.

Call:

To bet an equal amount as the previous player in real terms (in cases of a raise).

Check:

When a player chooses not to act (i.e. bet or fold) putting the action on the next player. This is only possible if there have been no previous bets (in that round).

Dealer:

The player or person who deals (distributes) the cards in a game of poker. It is often an advantage to be dealer as he is the ultimate person to act, allowing him to gain information before choosing an action (i.e. to fold, bet, call, or raise).

Directed Graph:

A directed graph is used when representing an extensive game. The vertices represent different positions in the game and ultimate payoffs.

Dominated Strategy:

A dominated strategy is a pure strategy, which involves a worse payoff for all possible corresponding actions of the opposing player than another strategy.

End Game:

The end game in, all forms of, poker often refers to the last rounds of betting where no further cards are introduced, hence each players hand is determined.

Extensive form:

The extensive form of a game is represented by a tree, indicating the nature of the game, information sets, positions, sequential moves, and *payoffs*.

Hand:

A hand is the combination of cards that a player has at a point in the game.

Maximal player:

A maximal player will exploit any observed weaknesses in an un-optimal player.

Mixed strategy:

A mixed strategy arises when there is no single pure dominant strategy available, thus a mix of different pure strategies must be employed at different probabilities, making the opposing player indifferent; subsequently resulting in an optimal expected value against an optimal player.

Optimal player:

An optimal player will play following an optimal mixed strategy making him indifferent to any actions of his opponent, however this will only be effective, or more importantly, necessary when playing against an optimal player.

Payoff/Expected value:

A payoff represents the result of an action or expected earnings of a strategy, which will be achieved in the long run.

Perfect information:

This occurs when a player is aware of all past moves and positions of both himself and his opponent. He can thus distinguish between singular information sets.

Player:

A player is a person or agent involved in a game making decisions and actions

Raise:

A raise occurs when a player chooses to bet an amount greater than the previous bet.

Rational player:

A rational player will appropriately – try to – choose when to play optimally or maximally and will avoid any perverse or inadmissible moves (i.e. folding with an optimal hand).

Showdown:

This occurs when each player has placed an equal amount into the pot and neither has folded. The players show their hands; the one with the hand of highest value wins the pot.

Strategic form:

This is also known as the normal form of a game, which represents a game in a matrix showing the payoffs of pure, admissible, strategies. However it fails to show any information sets of the role of sequential, as opposed to, simultaneous moves.

Strategy:

A strategy can either be mixed or pure. A pure strategy is essentially one possible set of actions of a player.

Zero-sum game:

A zero-sum game occurs when the sum of the pay-offs of each player is equal to zero. One player's loss is another player's gain (in two-person zero-sum games).

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