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Extended Essay

The $3n + 1$ Conjecture: Behaviour of the Stopping Time Function

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Abstract

Pick any natural number n. If it is even, divide it by two; if it is odd, multiply it by three and add one to make it even, and then divide the resulting number by two. Take the number you obtained and repeat this process, again and again.

The $3n+1$ Conjecture states that no matter what positive integer you start out with, you will eventually reach 1 and then fall into the trivial cycle 2 – 1.

The proof of this statement is a longstanding unsolved problem in number theory. Many mathematicians have tried to assail it but none have succeeded, prompting mathematician Paul Erdős to declare that "*Mathematics is not yet ready for such problems.*"

Nevertheless, because of the contrast between the problem's apparent simplicity and the underlying difficulty in making any progress towards finding a solution, many people find it captivating and have investigated various aspects of the function.

Thus, I shall quickly outline this deceptively simple question that has stumped mathematicians for decades and then raise an intriguing question which immediately arises from the study of the $3n+1$ iterative map: "*how many times does the aforementioned process need to be repeated on a given starting number for it to reach 1?*"

This interrogation leads to the investigation of the so-called total stopping time function, denoted by $\sigma(n)$, which I define and further discuss in the essay. Finally, I report on the findings of my examination of the patterns it exhibits, and provide an explanation for the phenomenon I observed, based on the accumulation of coalescences congruent modulo $2^{\sigma(s)-1}$.

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Alex Lupsasca

Table of Contents

Alex Lupsasca

Statement of the Problem

In mathematical notation, the Collatz function $T(n)$ is defined as:

$$T(n)=\begin{cases}\frac{n}{2} & \text{for } n\equiv 0 \bmod 2\\ \frac{3n+1}{2} & \text{for } n\equiv 1 \bmod 2\end{cases}$$

For a given starting integer n, which shall henceforth be referred to as the "**flight number**," the iteration of this function produces an infinite Collatz sequence $\{n, T(n), T^{(2)}(n), T^{(3)}(n), T^{(4)}(n)\ldots\}$.

The numbers that such a sequence contains are termed "**altitudes**" of the flight n and form what is called the "**forward orbit**," or "**trajectory**," of that flight.

For instance, flight number 11 has the following trajectory:

$$11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1, 2, 1, 2, 1\ldots$$

As another example, the forward orbit of flight 15 is:

$$15, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1, 2, 1, 2, 1\ldots$$

It is obvious from these examples that if a flight ever reaches the altitude 8, it will obligatorily fall to 4 and then into the never-ending cycle 2 – 1. But do all the starting values of n generate trajectories which fly through 8 and thus end up alternating between 2 and 1 *ad infinitum*?

The Collatz conjecture answers this question affirmatively. It states that:

"The process of iterating the Collatz function yields orbits which eventually reach the cycle 2 – 1, regardless of the positive integer initially chosen."

More formally, this is expressed as:

$$\forall n \in N > 0\ \exists i \in N : (a_0 = n \Rightarrow a_i = 1)$$

If the conjecture is false, it can only be because there is some natural number which gives rise to a sequence that does not comprise the altitude 1. Such an orbit might enter a repeating cycle that excludes 1 or increase without bound.

If the conjecture is true, an interesting aspect of the problem that lends itself to investigation is the "**total stopping time function**," which I shall introduce and formally develop over the course of this essay in order to address the question:

"*How many iterations of the function are necessary for a given number to become imprisoned in the cycle 2 – 1?*"

My enquiry into the problem was to create a program that would compute the total stopping times of a wide range of starting values n (i.e. the number of altitudes comprised by their forward orbits until they first reach 1) and hence try to find patterns in the behaviour of the stopping time function, denoted by $\sigma(n)$.

For very high values considered, a striking trend does indeed become apparent: more and more consecutive values have the same stopping time, and thus a small number of stopping times reappear with increasing frequency.

To facilitate manipulation of the function, the conditional form can be condensed into a single formula. There are various possible ways to accomplish this condensation but arguably the most elegant one can be found in the work of Marc Chamberland **[3],** whose method uses elementary trigonometric functions.

Indeed, consider an integer n:

- If n is odd, then $\cos\left(\frac{\pi}{2}n\right)$ is congruent modulo 2π to either $\cos\left(\frac{\pi}{2}\right)$ or $\cos\left(\frac{3\pi}{2}\right)$. As such, $\cos\left(\frac{\pi}{2}n\right)=0$ and thus $\cos^2\left(\frac{\pi}{2}n\right)=0$. Similarly, $\sin\left(\frac{\pi}{2}n\right)$ is congruent modulo 2π to either $\sin\left(\frac{\pi}{2}\right)$ or $\sin\left(\frac{3\pi}{2}\right)$ and therefore $\sin\left(\frac{\pi}{2}n\right)=\pm 1$. As a result, $\sin^2\left(\frac{\pi}{2}n\right)=1$.

- If n is even, then $\cos\left(\frac{\pi}{2}n\right)$ is congruent modulo 2π to either $\cos\pi$ or $\cos 2\pi$ and $\cos\left(\frac{\pi}{2}n\right)=\pm 1$. Similarly, $\sin\left(\frac{\pi}{2}n\right)$ is congruent modulo 2π to either $\sin\pi$ or $\sin 2\pi$ and therefore $\sin\left(\frac{\pi}{2}n\right)=0$. As such, $\cos^2\left(\frac{\pi}{2}n\right)=1$ and $\sin^2\left(\frac{\pi}{2}n\right)=0$.

Therefore, the function $T(n)$ can be expressed as:

$$T(n)=\frac{3n+1}{2}\sin^2\left(\frac{\pi}{2}n\right)+\frac{n}{2}\cos^2\left(\frac{\pi}{2}n\right)$$

Using the following trigonometrical identities:

$$\cos^2 A=\frac{1+\cos 2A}{2}$$
$$\sin^2 A=\frac{1-\cos 2A}{2}$$

- 3 -

The function can be simplified:

$$T(n)=\left(\frac{3n+1}{2}\right)\left(\frac{1-\cos n\pi}{2}\right)+\left(\frac{n}{2}\right)\left(\frac{1+\cos n\pi}{2}\right)$$

$$T(n)=\frac{1}{4}(3n-3n\cos n\pi+1-\cos n\pi+n+n\cos n\pi)$$

$$T(n)=\frac{1}{4}(4n+1-2n\cos n\pi-\cos n\pi)$$

$$T(n)=\frac{1}{4}[4n+1-(2n+1)\cos n\pi]$$

$$T(n)=n+\frac{1}{4}-\frac{2n+1}{4}\cos n\pi$$

This trigonometric interpolation is an extension of the Collatz function to the real line. However, as this essay is only concerned with the positive integers, I can rewrite the formula in a simpler form by making use of the fact that $\cos n\pi$ equals 1 for even values and -1 for odd values of n and thus:

$$T(n)=n+\frac{1}{4}-\frac{2n+1}{4}(-1)^n$$

From this formula, I have succeeded in deriving a relationship which any two consecutive altitudes a_n and a_{n+1} within any Collatz sequence must satisfy:

$$4a_{n+1}^{\ 2}-2a_{n+1}-8a_{n+1}a_n+a_n+3a_n^{\ 2}=0$$

I have called this formula the sequence-generating equation and shall refer to it later when I make use of it in one of my approaches to the problem.

Proof of the relationship:

$$a_{n+1} = a_n + \frac{1}{4} + \frac{2a_n + 1}{4}\cos(a_n \pi)$$

$$a_{n+1} = a_n + \frac{1}{4} + \frac{2a_n + 1}{4}(-1)^{a_n}$$

$$a_{n+1} - a_n - \frac{1}{4} = \frac{2a_n + 1}{4}(-1)^{a_n}$$

$$\left(a_{n+1} - a_n - \frac{1}{4}\right)^2 = \left(\frac{2a_n + 1}{4}\right)^2$$

$$a_{n+1}^{\ 2} - \frac{a_{n+1}}{2} - 2a_{n+1}a_n + a_n^{\ 2} + \frac{a_n}{2} + \frac{1}{16} = \frac{4a_n^{\ 2} + 4a_n + 1}{16}$$

$$4a_{n+1}^{\ 2} - 2a_{n+1} - 8a_{n+1}a_n + 4a_n^{\ 2} + 2a_n + \frac{1}{4} = a_n^{\ 2} + a_n + \frac{1}{4}$$

$$4a_{n+1}^{\ 2} - 2a_{n+1} - 8a_{n+1}a_n + 3a_n^{\ 2} + a_n = 0$$

Origins of the Conjecture

The $3n+1$ Conjecture has circulated by word of mouth in the mathematical community for many years and its exact origin is consequently obscure.

Traditionally, the problem is attributed to Lothar Collatz (1910 – 1990), a German mathematician who taught at the University of Hamburg and spent his youth (as well as the better part of his life) trying to discover a proof, to no avail. It is widely believed that he first formulated the conjecture as a student in 1937 and, as a result, it usually bears his name.

At some point during World War II, Stanislaw Ulam, a researcher at Los Alamos, began to investigate the conjecture in his free time but failed to formulate a proof. Instead, he described it to his friends who went on to call it **Ulam's problem**.

However, it is another mathematician from the University of Hamburg who spread the problem around the world: Helmut Hasse one day stumbled upon Collatz' work and his interest was peaked by this seemingly simple question. He consequently began to give lectures on this topic, and it became known as **Hasse's algorithm**. Because he presented it at the University of Syracuse, USA, in 1950, it is also called (most notably in the French speaking world) the **Syracuse conjecture**. Simultaneously, British mathematician Bryan Thwaites independently discovered the problem in 1952 and it was therefore renamed by the English **Thwaites' conjecture**.

Later, in the 1960s, Shizuo Kakutani lectured on the topic at Yale University and the University of Chicago, and proving the conjecture immediately became **Kakutani's problem**. Some attentive members of his audience remarked that the number sequences jumped up and down like hailstones in a cloud before ultimately plummeting to 1 and dubbed it the **Hailstone problem**.

Kakutani's presentations and discussions encouraged mathematicians to research the question and for the following months, professors, assistants as well as students from these two universities would feverishly attempt to make any sort of progress towards proving the conjecture.

As proof eluded everyone, a joke was made that the KGB had intentionally set forth the question to slow down the progress of American mathematics **[2]**.

Flight Time

As we have seen from the forward orbits of flights 11 and 15, Collatz sequences seem to follow erratic trajectories which jump around before eventually settling into the cycle 2 – 1.

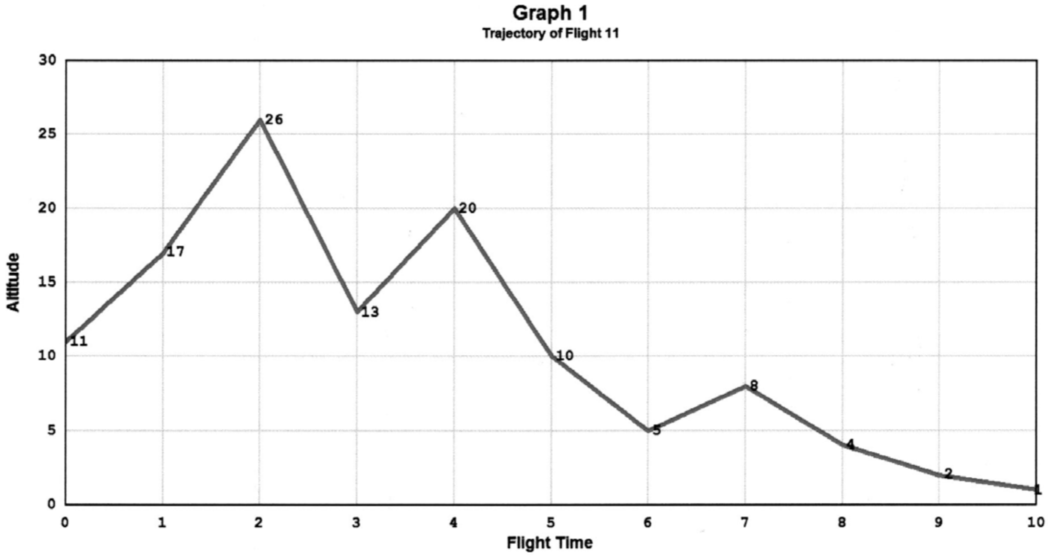
Immediately, one is compelled to ask: "*How much time does this process take*"

In other words, how many times must the function be iterated for a sequence to reach 1? How does the length of a trajectory vary for different flight numbers?

These questions lead to the introduction of the concept of "**flight time**," which corresponds to the number of altitudes that the forward orbit of a given flight comprises before it reaches 1. Hence the "**total stopping time function**," denoted by $\sigma(n)$, can be defined as the function that assigns to every natural number n its flight time. Recall the Collatz sequence generated by flight 11:

$$11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1\ldots$$

As there are 10 steps in this sequence before it converges to the cycle 2 – 1, the flight time of 11 is 10 and thus $\sigma(11) = 10$. For any flight, this can be illustrated on a graph such as this one:



A Heuristic Approach

A **heuristic argument** is a method or principle that has been shown by experimental (especially trial-and-error) investigation to be a useful aid in learning, discovery and problem-solving. Although the conjecture has not been proven, a convincing heuristic argument has been formulated to support it **[4]**.

At first glance, one could reason in the following way. Any number is either even or odd. Since there are as many even integers as there are odd integers, then one half of the time, during a flight, the formula $3n+1$ will be used and, the other half, a number will be halved. On average, then, a number will be multiplied by 3/2 after any two iterations of the function, and therefore all flights will keep reaching higher and higher altitudes, rendering the conjecture false.

However, this argument is clearly naïve because it fails to take into account the fact that when the $3n+1$ formula is applied to an odd integer, it becomes even.

Consider now the following argument. Choose an odd integer n_0 at random and iterate the function on it until the next odd integer n_1 is obtained.

One fourth of the positive integers are divisible by $4 = 2^2$ and can thus be halved twice consecutively; one eighth are divisible by $8 = 2^3$ and can be halved three times repeatedly; one sixteenth are divisible by $16 = 2^4$ and can be halved 4 times consecutively, etc.

Then, clearly, one half of the time $n_1 = \frac{3n_0 + 1}{2}$; one quarter of the time $n_1 = \frac{3n_0 + 1}{4}$; one eighth of the time $n_1 = \frac{3n_0 + 1}{8}$ and so on. Thus, the expected growth between two odd integers in a flight is given by the multiplicative factor:

$$C = \left(\frac{3}{2}\right)^{\frac{1}{2}} \left(\frac{3}{4}\right)^{\frac{1}{4}} \left(\frac{3}{8}\right)^{\frac{1}{8}} \cdots = \frac{3}{4}$$

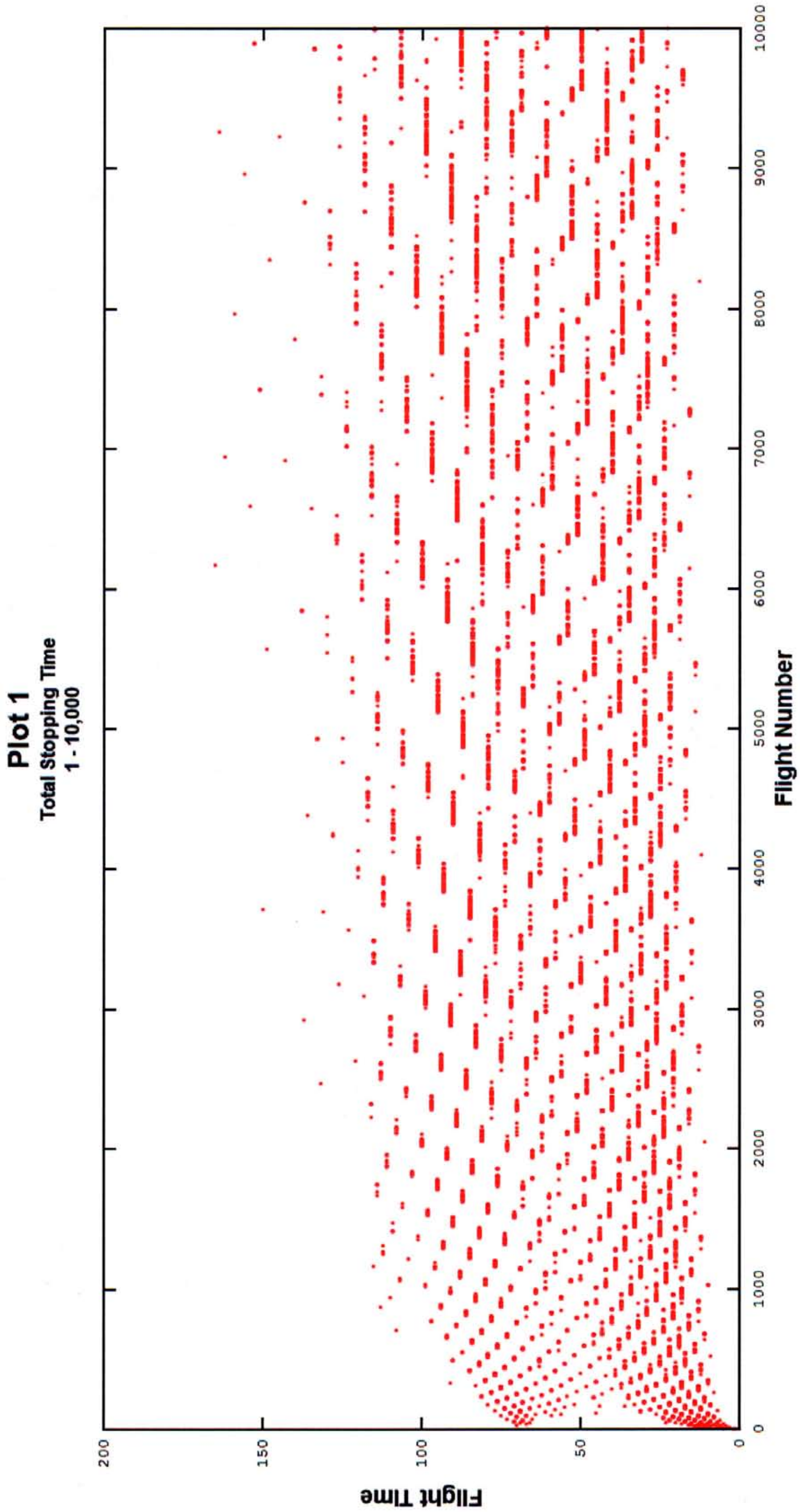
On average then, the flight reduces its altitude by 25% when going from an odd number to the next. This indicates that all flights should thus reach one and supports the conjecture.

A Computational Approach

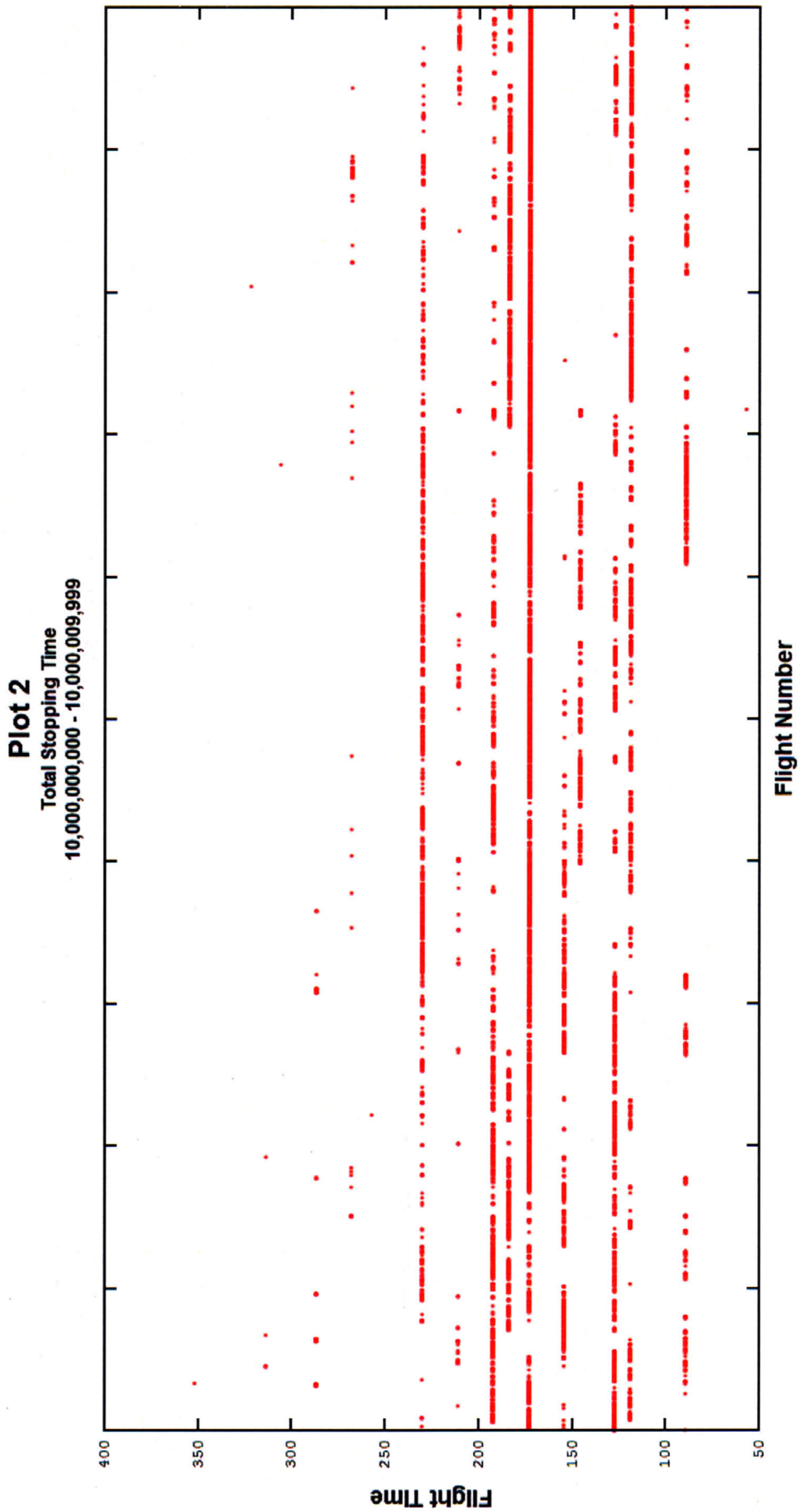
In order to gather data about the stopping time function and familiarize myself with it, I wrote a program to compute $\sigma(n)$ for values of n ranging from 1 to 10^{15}.

I then generated plots of this function for intervals of 10,000 integers at different order of magnitudes. Plot 1 represents the total stopping time function for the first 10,000 flights. It is very difficult to perceive a pattern in the flight times, as they vary greatly. However, as the starting numbers considered grow larger, the flight times vary less and less within an equal interval of 10,000 flights. This can be checked by looking at Plot 2, which plots the stopping times of flights around 10^{10}.

To verify this trend, I observed a multitude of intervals and found that the larger the flight numbers are, the more the erratic dots group together to form lines. Due to software constraints, the furthest I could examine the function is in the whereabouts of 10^{15}. Plot 3 shows the stopping times of flights in this interval, where the trend I have remarked upon is glaringly obvious.

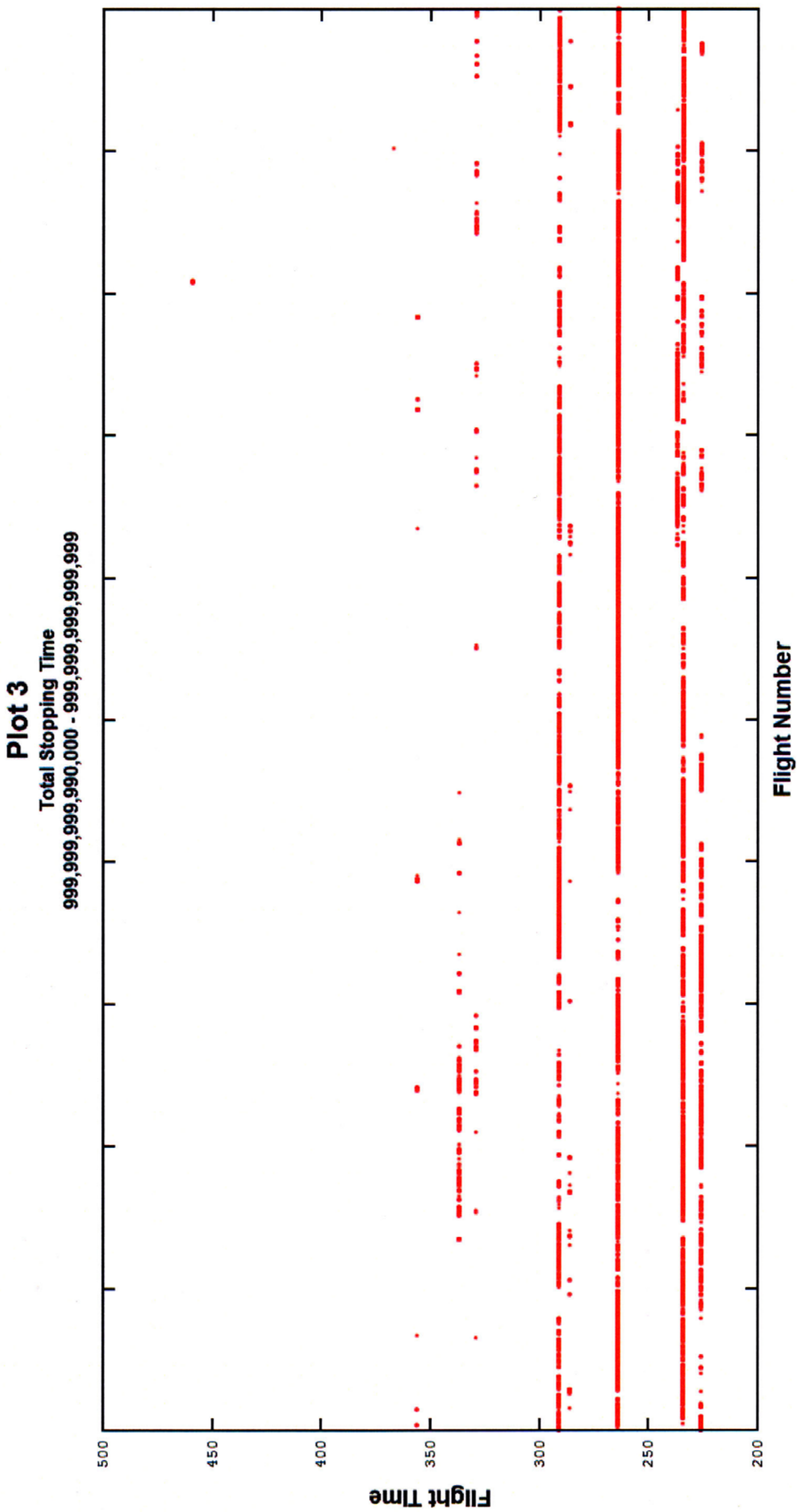

Plot 1
Total Stopping Time
1 - 10,000
Flight Number
Flight Time
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000
0
50
100
150
200

Alex Lupsasca



- 12 -

Alex Lupsasca



- 13 -

Observation of the Pattern

The appearance of lines on the graph of $\sigma(n)$ means that for an increasingly numerous amount of consecutive integers the total stopping time is the same, a fact which I initially found to be very peculiar.

A simple analysis of the data can confirm this: in the range 1 - 10000, there are 224 different possible flight times; in the range 10,000,000,000 - 10,000,009,999 there are only 18; finally and most impressively, in the range 999,999,999,990,000 - 999,999,999,999,999 the flight times take only 11 different values, as can be seen on the following table:

Flight Time	226	234	237	264	286	291	329	337	356	367	459
Frequency of this value	886	2390	454	3648	64	2181	136	195	40	1	5

Table of the frequencies of recurring flight times around 10^{15}

Explanation of the Phenomenon

In order to confirm an intuition I had, I wrote another program that would compute the highest altitude in the trajectory of a flight. To use the example of flight 11 yet again, Graph 1 clearly reveals that the maximum altitude is 26.

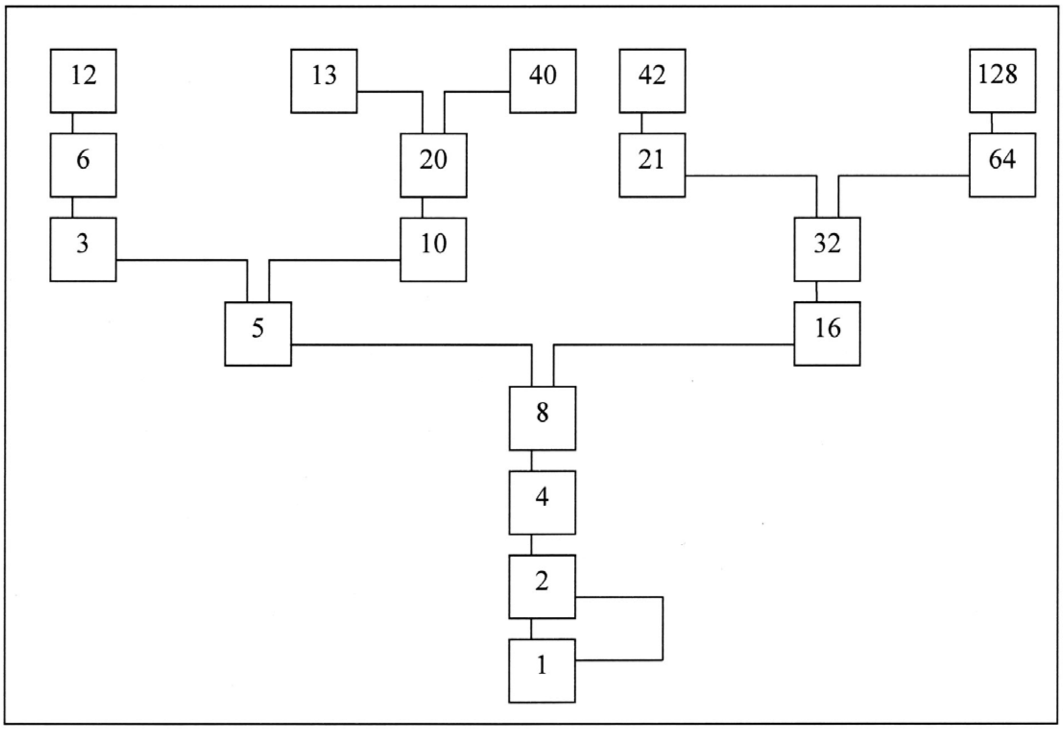
I noticed that many consecutive flights that had the same total stopping time also shared the same maximum altitude. For example, flights 54 and 55 both have a flight time of 71 iterations. They also share the same maximum altitude of 4616.

It thus became apparent to me that at some point, the trajectories of these flights must "**coalesce**." Indeed, if all sequences really do converge to 1, their trajectory must inevitably contain the altitudes 2, 4 and 8 (as solving the sequence-generating equation by setting $a_{n+1} = 8$ will reveal).Therefore, in order to reach 1, all orbits must join each other at some value, the last of which is 8.

In fact, to reuse the analogy of the airplane, the attractive cycle 2 – 1 is like a small airport with one "**landing strip**" in the sense that flights can come from anywhere in the realm of the natural numbers but must then rejoin one of the limited number of approaching routes (and eventually touch down on the unique landing strip 8 – 4 – 2 – 1).

This idea of examining the Collatz function backwards, starting at 1, leads to the creation of an "**inverted Collatz tree**" (easily generated by solving the sequence- generating equation backwards with a_{n+1} initially equal to 1) which clearly shows that trajectories must eventually coalesce if the conjecture is to be true since there is a limited number of branches connected to 1.

For example, as the tree shows, flights 12 and 13 both have the same flight time (namely 7 iterations of the Collatz function) and, although they are located on different branches, they both join the same forward orbit after 3 steps because there is a rarefaction of possible paths the lower one descends in the tree.



Inverted Collatz Tree (first 7 levels)

This idea is supported by the following example. Let us examine iterations of $T(n)$ on numbers of the form $8k+4$ and $8k+5$ for all positive k:

- $8k+4$ is even, so we first divide it by two to obtain $4k+2$, which we again halve to reach $2k+1$. An additional iteration yields $3k+2$.
- $8k+5$ is odd, so we first multiply by 3 and add 1, and then divide to obtain $12k+8$, which we halve once to get $6k+4$ and then again to obtain $3k+2$.

Therefore, the orbits of all the pairs of numbers of the forms $8k+4$ and $8k+5$ coalesce after 3 iterations to $3k+2$.

By setting $k=1$, this property of the Collatz function dictates that flights 12 and 13 coalesce after 3 steps upon reaching the altitude 5, and thereby confirms the validity of the inverted Collatz tree.

"*Is there a deeper underlying reason for this property?*" It is now time to explain the existence of the straight lines that emerge on the plots of $\sigma(n)$ and understand how exactly consecutive values of integers have the same flight time. Thus, I shall now present an argument initially put forth by Jeffrey Lagarias **[1]** and which I have further developed.

Suppose a and b are such that $\sigma(a) \equiv \sigma(b)(\bmod 2)$ and $\sigma(a) = s \geq \sigma(b) = t$. Then, clearly, the forward orbits of a and b coalesce after at most $s-1$ iterations as $T^{(s-1)}(a) = T^{(s-1)}(b) = 2$ since the orbit of b continues to oscillate between 2 and 1.

Now, let the parity function $\nu_k(n)$ be defined as:

$$\nu_k(n) = x_0(n) + x_1(n) + x_2(n) + x_3(n) + \ldots + x_{k-1}(n)$$

Where

$$T^{(i)}(n) \equiv x_{(i)}(n)(\bmod 2)$$

36 and 37 have the same flight time $\sigma(36) = \sigma(37) = 15$. Their trajectories are:

- (36, 18, 9, 14, 7, 11, 17, 26, 13, 10, 20, 5, 8, 4, 2, 1)
- (37, 56, 28, 14, 7, 11, 17, 26, 13, 10, 20, 5, 8, 4, 2, 1)

Clearly, their orbits coalesce after 3 iterations. It is interesting to look at the parity function of 36 and 37:

$$\nu_{16}(36) = 0+0+1+0+1+1+1+0+1+0+0+1+0+0+0+1 = 7$$

$$\nu_{16}(37) = 1+0+0+0+1+1+1+0+1+0+0+1+0+0+0+1 = 7$$

The parity function yields the same value for 36 and 37. This stands to reason as we would expect most consecutive numbers to need approximately the same number of halving steps and of "multiplying by 3" steps in order to reach 1 because they have similar sizes. If that is indeed the case, then the trajectories of $2^{s-1}k+a$ and $2^{s-1}k+b$ coalesce after at most $s-1$ iterations, for $k \geq 0$.

More precisely, $\sigma(2^{s-1}k+a) = \sigma(2^{s-1}k+b)$ then holds for $k \geq 1$ and thus, the original coalescence of a and b produces an infinite arithmetic progression $(\mathrm{mod}\,2^{s-1})$ of coalescences. The perpetual accumulation of these coalescences of numbers close together in size leads to the phenomena investigated.

The numbers 36 and 37 satisfy all these conditions and have $s-1 = 15-1 = 14$. Thus all numbers of the form $2^{14}k+36$ and $2^{14}k+37$ should have the same stopping time. For $k=1$ we find that, indeed, $\sigma(2^{14}+36) = \sigma(2^{14}+37) = 103$. For $k=2$, we also find that $\sigma(2^{15}+36) = \sigma(2^{15}+37) = 77$ and so on.

Let us look at the numbers 4 and 5. Their flight numbers are 2 and 4 respectively, so they are congruent modulo 2, and $s-1 = \sigma(5)-1 = 3$.

Thus, trajectories of the numbers 2^3k+4 and 2^3k+5 coalesce after at most 3 iterations. This therefore explains our earlier observation that all numbers of the form $8k+4$ and $8k+5$ rejoin the same forward orbit after three steps.

If we set $k=4$ then, if the argument is correct, the Collatz sequences of $8\times4+4=36$ and $8\times4+5=37$ coalesce after 3 iterations. As we have seen above, that is indeed the case.

Conclusion

The argument I have detailed above provides a satisfactory explanation for the trend I detected in the behaviour of the stopping time function. However, my explanation does comprise many limitations: most glaringly, it does not provide any indication about the rate at which the coalescences appear and accumulate.

In the future, it would be very interesting to take this research further and determine, for instance, a law that governs the appearance of new coalescences. In the interval of 10,000 values around 10^{10} there are 49 repeating flight times, whereas in a similar interval around 10^{15} there are only 11. It would be useful to be able to predict the rate of this rarefaction and thence find a way of determining the order of magnitude at which one should examine an interval in order to find a given number of recurring values of $\sigma(n)$.

Also, one might want to ask: "*Does this trend go on forever?*" That seems unlikely as it would suggest that the graph of the stopping time tends to a unique line, which would be very surprising. Nonetheless, further investigation in the domain would undoubtedly prove to be fruitful.

Either it's true or it isn't!

A first glance at the problem might not reveal much about the veracity of the conjecture, but I believe an in-depth exploration and familiarisation with the function will convince anyone that the conjecture is indeed true. I have personally checked it up to 10^{15} and an ongoing distributed computing project of which I am part of has gone up to 2^{65}. This, combined with the heuristic argument I have presented and the behaviour of the stopping time function (which tends to form a "line"), have led me, as well as many professional mathematicians, to strongly believe in its validity.

Of course, belief is not sufficient and it is necessary not to relent in attempts to ascertain the truth of the conjecture and address this burning question with absolute certainty. I hope that as time passes the Collatz Conjecture will gain in prestige, allure and mystery, and thereby attract more researchers, just as the combination of a simple question with decades of stumped mathematician fuelled the quest for the proof of Fermat's Last theorem.

This problem looks easy! Why has no one solved it yet?

Could the problem be intractable? When looking at generalisations of the conjecture which consider iterations of the form $mn+x$, most combinations of m and n seem to generate divergent sequences and many others are probably unsolvable. For instance, the $5n+1$ sequence (where we multiply by 5 instead of 3 when a number is odd) seems to produce an infinitely long flight, although it is very difficult to prove. Beyond the $3n+1$ conjecture lies a new world of research which prompts amateursto continue working at the problem, regardless of its difficulty (or impossibility), until the conjecture is either proved or disproved. In the meantime, many findings remain to be made.

I hope I have demonstrated how fascinating the $3n+1$ conjecture is and that I have succeeded in raising your interest high enough for you to want to try your hand at attacking the problem. I must warn you, however, that many bright minds have tried but none ever triumphed, prompting even famous mathematician Paul Erdős to declare:

"*Mathematics is not yet ready for such problems.*"

Nonetheless, research is still ongoing as many amateur mathematicians regularly announce pseudo-proofs, including a very convincing hoax which appeared on the internet a few years ago.

The apparent difficulty of the $3n+1$ conjecture seemingly stems from the fact that it deterministically generates a "random" behaviour. Thus, mathematicians are confronted with a problem. On the one hand, the problem has structure, to the extent that it can be analysed as I have done, yet it is specifically this structure which prevents us from proving it is "random." On the other hand, considering the extent to which the problem is structureless and "random," it is difficult to analyse and even more difficult to rigorously prove anything. Therein, perhaps, lies its beauty.

Alex Lupsasca

Bibliography

[1] Jeffrey C. Lagarias: *The 3x+1 problem and its generalizations*, American Mathematical Monthly Volume 92, 1985.

[2] Günther J. Wirsching: *The Dynamical System Generated by the 3n+1 Function*, Lecture Notes in Mathematics, April 1998.

[3] Marc Chamberland: An Update on the 3x+1 Problem, , 2003.

[4] Jean-Paul Delahaye: *La conjecture de Syracuse*, Pour La Science N°247, May 1998.

Acknowledgments

Mark Ballantyne – Thank you for introducing me to the problem and encouraging me in my mathematical endeavours.

Irene Stewart – Thank you for the patience and effort you put in assisting me to write this essay.

Appendix

Source codes of Visual Basic programs related to the 3x+1 conjecture which can easily be implemented in Microsoft Excel:

- A program that computes the total flight time:

```
Function TotalFlightTime(x)
s = 0
1:
If Int(x / 2) = (x / 2) Then
x = (x / 2)
s = s + 1
GoTo 2
Else
x = (3 * x + 1) / 2
s = s + 1
GoTo 1
End If
2:
If x = 1 Then
GoTo 3
Else
GoTo 1
End If
3:
TotalFlightTime = s
End Function
```

- A program that computes the flight time in altitude:

```
Function AltitudeFlightTime(x)
s = 0
e = x
1:
If Int(x / 2) = (x / 2) Then
x = (x / 2)
GoTo 2
Else
x = (3 * x + 1) / 2
s = s + 1
GoTo 1
End If
2:
If x < e Then
GoTo 3
Else
s = s + 1
```

```
GoTo 1
End If
3:
AltitudeFlightTime = s
End Function
```

- A program that computes the maximum altitude reached:

```
Function MaxAltitude(x)
h = x
1:
If Int(x / 2) = (x / 2) Then
x = (x / 2)
GoTo 2
Else
x = (3 * x + 1) / 2
If h < x Then
h = x
End If
GoTo 1
End If
2:
If x = 1 Then
GoTo 3
Else
GoTo 1
End If
3:
MaxAltitude = h
End Function
```

The function $T(n) = n + \frac{1}{4} - \frac{2n+1}{4}\cos n\pi$ does not involve the concept of parity and allows for the conjecture to be extended to the real numbers and even the complex numbers. Iteration of that function will then produce a fractal which, although it is far beyond the scope of this essay, I wish to include for its aesthetical qualities: